The Semi-Classical Relativistic Darwin Potential for Spinning Particles in the Rest-Frame Instant Form: 2-Body Bound States with Spin 1/2 Constituents.

David Alba

Dipartimento di Fisica

Universita' di Firenze

L.go E.Fermi 2 (Arcetri)

50125 Firenze, Italy

E-mail: ALBA@FI.INFN.IT

and

Horace Crater

The University of Tennessee Space Institute

Tullahoma, Tennessee

37388 USA

E-mail: HCrater@utsi.edu

and

Luca Lusanna

Sezione INFN di Firenze

L.go E.Fermi 2 (Arcetri)

50125 Firenze, Italy

E-mail: LUSANNA@FI.INFN.IT

Abstract

We extend previous results on the extraction of the Darwin potential to all orders in c^{-2} from the radiation gauge Lienard-Wiechert solution for the system of N positive-energy scalar particles plus the electromagnetic field in the Wigner-covariant rest-frame instant form of dynamics to the case of N positive-energy spinning particles. This is done in the semi-classical approximation of using *Grassmann-valued electric charges* for *regularizing* the Coulomb self-energies and *extracting* the *unique semi-classical* action-at-a-distance interaction hidden in any Green function used for the Lienard-Wiechert solution. By describing semi-classically also the *spin* of the particles with Grassmann-variables, by means of a semi-classical Foldy-Wouthuysen transformation applied the the Dirac-like constraints of the manifestly Lorentz covariant spinning particles, we determine the coupling of positive-energy spinning particles to the electric field in the semi-classical approximation. Then we follow the same procedure developed for scalar particles and, in the sector where there is no *in*-radiation, we determine the effective semi-classical interparticle potential. Besides the relativistic *Darwin* term there are *spin-orbit* and *spin-spin* terms in the potential. Quantization of the lowest order (in c^{-2}) part of the closed form of the effective Hamiltonian in the case N=2 reproduces *exactly* the standard result of the reduction of the *Bethe-Salpeter equation* for the bound states of two spin 1/2 constituents of arbitrary mass (hydrogen atom, positronium, muonium).

March 20, 2001

Typeset using REVTEX

I. INTRODUCTION.

At the classical level, the accepted mathematical description of the four basic interactions starts with an action principle. Manifest Lorentz invariance and local gauge invariance force these interactions to make use of *singular Lagrangians*. This implies that their Hamiltonian description must rely on the *Dirac-Bergmann theory of constraints* [1,2]. Constraint dynamics provides a natural formalism for any program of a unified description of these interactions in terms of *Dirac-Bergmann observables*. Those observables are gauge invariant and deterministic variables which describe a canonical basis of measurable quantities. When one begins with a singular Lagrangian, the canonical momenta $p_i = \partial L/\partial \dot{q}_i$ are not independent. The relations among them, $\phi_\alpha(q,p) \approx 0$, are called *primary* constraints (the weak inequality $\approx$ means that the equality sign cannot be used inside Poisson brackets). The canonical Hamiltonian $H_c(q,p)$ has to be replaced by the Dirac Hamiltonian $H_D = H_c + \sum_\alpha \lambda_\alpha(t)\phi_a$ which, by way of the arbitrary Dirac multipliers, then accounts for the restriction to the submanifold defined by the constraints. The time constancy of the primary constraints, $\partial_t\phi_\alpha = \{\phi_\alpha, H_D\} \approx 0$, either produces *secondary* Hamiltonian constraints or determines some of the Lagrange multipliers. This procedure is repeated for the secondary constraints and again if necessary (this is the *Dirac-Bergmann algorithm*) [3]. When the procedure is finished, there is a final set of constraints, $\chi_\alpha \approx 0$, defining the final submanifold on which the dynamics is consistently restricted, and a final Dirac Hamiltonian with a reduced set of arbitrary Dirac multipliers describing the remaining arbitrariness of the time evolution. Dirac divides the constraints into two subgroups: i) the *first class* constraints $\chi^{(1)}_\alpha \approx 0$ having weakly zero Poisson brackets with all constraints and being the generators of the *Hamiltonian gauge transformations* of the theory and ii) the *second class* constraints $\chi^{(2)}_\alpha \approx 0$ [1] with $\det(\{\chi^{(2)}_{n_1}, \chi^{(2)}_{n_2}\}) \neq 0$, corresponding to pairs of *inessential* variables that can be eliminated. The vehicle that Dirac invented to eliminate, in a systematic and symmetrical way, from the dynamical equations those variables constrained by the second class constraints are known as *Dirac brackets*. Second class constraints may be primary or secondary constraints but they can also be introduced by hand. This can be done in two different ways. Firstly one can explicitly break the gauge freedom by the introduction of gauge fixing conditions on the gauge variables, so that the original first class constraints and the gauge fixing ones become a system of second class constraints. Secondly one can introduce pairs of solutions to the equations of motions (for example, field equations whose solutions express fields in terms of canonical variables for particles serving as sources for the fields). In general this procedure may break the original manifest Lorentz covariance of the theory (but maintaining Wigner covariance, when we work in the *rest-frame instant form of dynamics* [4]).

If the end aim is a unified description of the four interactions, then in the application of the Dirac-Bergmann algorithm to special relativistic theories it is advantageous to reformulate the theory in such a way as to allow a natural transition to the coupling of gravity. With this in mind, recent studies (see [4,6] and references cited therein) have taken advan-

[1] They always appear in pairs when there are only bosonic degrees of freedom.

tage of work by Dirac [1] on general relativity arriving to a description of special relativistic systems on arbitrary spacelike hypersurfaces (*parametrized Minkowski theories*). Then, for every configuration of the system with timelike total 4-momentum the description can be restricted to the so-called *Wigner hyperplanes* orthogonal to the total 4-momentum. This is the intrinsic rest frame of the configuration. On it a new instant form of dynamics (the *Wigner-covariant rest-frame instant form*) can be introduced.

In a recent paper [5] we presented the technical completion of one segment of this program [6], that of N scalar charged particles plus the electromagnetic field when the charges of the particles are described by bilinears in Grassmann variables. There we analyzed how to extract the action-at-a-distance interparticle potential hidden in the semi-classical Lienard-Wiechert solution of the electromagnetic field equations, a subset of the solutions of the equations of motion for the isolated system formed by N positive-energy scalar charged particles plus the electromagnetic field. The problem is formulated in the Wigner-covariant rest-frame instant form of dynamics [4,6], which is defined on the Wigner hyperplanes orthogonal to the total time-like four-momentum of the isolated system and which requires the choice of the sign of the energy of the particles (we considered only positive energies)

This was possible due to the *semiclassical approximation* of using Grassmann-valued electric charges ($Q_i^2 = 0$, $Q_i Q_j \neq 0$ for $i \neq j$) as an alternative to the extended electron models used for the regularization of the Coulomb self energies. How this happens was shown in Ref. [4], where the Coulomb potential was extracted from the electromagnetic potential by making the canonical reduction of the electromagnetic gauge freedom via the Shanmugadhasan canonical transformation. This is equivalent to the use of a *Wigner-covariant radiation* (or Coulomb) *gauge* in the rest-frame instant form.

Ref. [7] presented the retarded Lienard-Wiechert solution for the transverse electromagnetic field in the rest-frame instant form radiation gauge: in this gauge, due to the transversality, the retarded Lienard-Wiechert potential associated with each charged particle depends on the whole past history of the other particles. At the semi-classical level a single accelerated charged positive-energy particle with Grassmann-valued electric charge does *not* radiate even if it has a non-trivial Lienard-Wiechert potential, avoiding therefore the acausal features of the Abraham-Lorentz-Dirac equations, and has *no* mass renormalization. However, a system of N charged particles produces, by virtue of the *interference terms* from the various retarded Lienard-Wiechert potentials of the particles, a radiation which reproduces the standard Larmor expression for radiation in the wave zone, when the particles are considered as *external* sources of the electromagnetic field and their equations of motion are *not* used.

If instead the particles are considered *dynamical*, the use of their equations of motion and of the semi-classical approximation leads to a drastic simplification of the Lienard-Wiechert potentials and fields. Indeed, if we make an equal time expansion of the delay by expressing these potentials and fields in terms of particle coordinates, velocities and accelerations of every order, it turns out that *all the accelerations decouple* at the semi-classical level due to the particle equations of motion. Therefore, at the semi-classical level the retarded, advanced and symmetric Lienard-Wiechert potentials and electric and magnetic fields coincide and *depend only* on the positions and velocities of the particles, so that we can find their phase space expression in terms of particle positions and momenta.

In this way the semi-classical Lienard-Wiechert potential and fields can be reinterpreted as *scalar and vector interparticle instantaneous action-at-a-distance potentials*. It is then possible to identify a semiclassical reduced phase space containing only particles by eliminating the electromagnetic field by adding by hand second class constraints which force the transverse potential and electric field canonical variables to coincide with the semi-classical Lienard-Wiechert ones in the absence of incoming radiation: $\vec{A}_{\perp}(\tau,\vec{\sigma}) - \vec{A}_{\perp LW}(\tau,\vec{\sigma}) \approx 0$, $\vec{\pi}_{\perp}(\tau,\vec{\sigma}) - \vec{\pi}_{\perp LW}(\tau,\vec{\sigma}) \approx 0$. Let us remark that this could be done also in presence of an arbitrary incoming radiation $\vec{A}_{\perp(rad)}(\tau,\vec{\sigma})$, $\vec{\pi}_{\perp(rad)}(\tau,\vec{\sigma}) = -\frac{\partial}{\partial\tau}\vec{A}_{\perp(rad)}(\tau,\vec{\sigma})$ [2] by modifying the constraints to the form $\vec{A}_{\perp}(\tau,\vec{\sigma}) - \vec{A}_{\perp LW}(\tau,\vec{\sigma}) - \vec{A}_{\perp(rad)}(\tau,\vec{\sigma}) \approx 0$, $\vec{\pi}_{\perp}(\tau,\vec{\sigma}) - \vec{\pi}_{\perp LW}(\tau,\vec{\sigma}) - \vec{\pi}_{\perp(rad)}(\tau,\vec{\sigma}) \approx 0$.

The reduced phase space is obtained by means of the introduction of the Dirac brackets associated with these second class constraints. Since the old particle positions and momenta are no longer canonical in this reduced phase space, we had to find the new (Darboux) basis of particle canonical variables. The generators of the *internal* Poincaré' group inside the Wigner hyperplanes in the rest-frame instant form of dynamics can be reexpressed in terms of these new variables: the 3-momentum $\vec{\mathcal{P}}_{(int)}$ and the angular momentum $\vec{\mathcal{J}}_{(int)}$ become equal to those for N free scalar particles (as expected in an instant form). The interaction dependent boosts $\vec{\mathcal{K}}_{(int)}$ are proportional to the *internal* canonical center of mass $\vec{q}_+$ inside the Wigner hyperplane: $\vec{q}_+ \approx 0$ are the gauge-fixings to be added to the rest-frame conditions $\vec{\mathcal{P}}_{(int)} \approx 0$, if one wishes to re-express the dynamics only in terms of particle *internal* relative variables. Therefore we have a unfaithful representation of the *internal* Poincaré group. Also the energy-momentum tensor has been evaluated in the new canonical variables and there is a suggestion on how to find the Møller center of energy of a cluster of n particles contained in the N particle isolated system.

The Hamiltonian in the rest frame instant form, generating the evolution in the rest-frame time of the decoupled *external* canonical center of mass, is the *internal* energy generator $M = \mathcal{P}^{\tau}_{(int)}$ (the invariant mass of the isolated N particle system). The semi-classical Lienard-Wiechert solution implies the existence of interparticle action-at-a-distance potentials of two types: *vector potentials*, minimally coupled to the Wigner spin 1 particle three-momentum, under the square root associated with the kinetic energies; ii) a *scalar potential* (including the Coulomb potential) outside the square roots. In the *semi-classical approximation* all these potentials can be replaced by a *unique scalar potential*, which is the sum of the *Coulomb potential* and of a *generalized Darwin one* for arbitrary N. It is the (semi-classical) non-static complete potential corresponding to the one photon exchange tree Feynman diagrams of scalar electrodynamics and is a completely new result. The expression we find contains no N-body forces, being simply a sum of two particle interactions. This is a consequence of our use of Grassmann charges and of the equal time description on the Wigner hyperplanes of the rest-frame instant form.

In the $N = 2$ case we obtain a closed form of the solution by evaluating it in the rest frame after the gauge fixing $\vec{q}_+ \approx 0$: the lowest order in $1/c^2$ contribution of the generalized Darwin potential agrees with the expression of the standard Darwin potential. We then

---

[2] It is an arbitrary solution of the homogeneous wave equation and must not be interpreted as a pair of canonical variables.

show that in a semi-classical sense a special solution of the Hamilton equations is the Schild solution [8] in which the two particles move in concentric circular orbits.

In this paper we will extend these results to positive-energy charged spinning particles, with both the electric charges and the spins described by Grassmann variables (pseudo- or semi-classical approximation), whose rest-frame instant form description was given in Ref. [9].

The resulting generalized Darwin potential will be the (semiclassical) static and non-static complete potential corresponding to the one-photon exchange tree Feynman diagrams of spinor electrodynamics, after the restriction to the subsector of positive-energy fermions.

The fact that the positive-energy spinning particles are a pseudo-classical description of positive-energy fermions immediately forces us to face a problem absent with scalar particles. In the manifestly Lorentz covariant approach, relativistic scalar particles are described by means of the mass-shell first class constraints

$$\chi_i = p_i^2 - m_i^2 \approx 0. \tag{1.1}$$

The positive- and negative-energy sheets of the mass hyperboloid correspond to the particle energies $p_i^0 \approx \pm\sqrt{m_i^2 + \vec{p}_i^2}$. When an *external* electromagnetic field is present we have

$$\chi_i = [p_i - Q_i A(x_i)]^2 - m_i^2 \approx 0, \tag{1.2}$$

so that $p_i^0 \approx Q_i A^0(x_i) \pm \sqrt{m_i^2 + [\,\vec{p}_i - Q_i \vec{A}(x_i)]^2}$. We see that at the classical level the positive- and negative-energy sheets of the mass hyperboloid of scalar particles never intersect even in presence of external electromagnetic fields [3]. Therefore, there is no problem in the rest-frame description of scalar particles with a fixed sign of the energy, since these particles will be always interpreted as the classical basis for Tomonaga-Schwinger asymptotic states.

On the other hand, the manifestly Lorentz covariant description of spinning particles requires the introduction of extra Dirac-like first class pseudo-classical constraints [11]

$$\chi_{Di} = p_{i\mu}\xi_i^\mu - m_i \xi_{i5} \approx 0, \tag{1.3}$$

whose quantization generates the one-particle Dirac equation [4] (for each particle we have $\xi_{i5} \mapsto \gamma_5$, $\xi_i^\mu \mapsto \gamma_5\gamma^\mu$, $\chi_{Di} \approx 0 \mapsto \gamma_5(p_{i\mu}\gamma^\mu - m_i)\psi(p_i) = 0$).

---

[3]As shown in Ref. [10], Appendix D, scalar particles give the semiclassical description only of those solutions of the Klein-Gordon equation coupled to an external electromagnetic field for which the Feshbach-Villar approach to the Klein-Gordon field allows a separation of positive-energies from negative-energies in an eikonal approximation. Otherwise the non-diagonalizability of the Feshbach-Villar Hamiltonian implies that generic solutions cannot be described only in terms of scalar particles when external fields are present, but require more elaborate quasi-particle concepts.

[4]The Grassmann variables ξ_i^μ, ξ_{i5} describe the *spin structure* and, after quantization, generate the Clifford algebra of Dirac matrices. We assume that the Grassmann variables describing the spin of different particles commute: $\xi_i^\mu \xi_j^\nu = \xi_j^\nu \xi_i^\mu$ for $i \neq j$, $\xi_i^\mu \xi_i^\nu + \xi_i^\nu \xi_i^\mu = 0$ and so on.

The mass-shell constraints $\chi_i = p_i^2 - m_i^2 \approx 0$ must be consistent with the pseudo-classical version $\{\chi_{Di}, \chi_{Dj}\} = i\delta_{ij}\chi_i$. In this sense *the Dirac equation is the square root of the Klein-Gordon one.*

When there is an *external* electromagnetic field, the Dirac-like constraints become

$$\chi_{Di} = [p_{i\mu} - Q_i A_\mu(x_i)]\xi_i^\mu - m_i \xi_{i5} \approx 0, \tag{1.4}$$

and the resulting mass-shell constraints

$$\chi_i = -i\{\chi_{Di}, \chi_{Di}\} = [p_i - Q_i A(x_i)]^2 - m_i^2 + iQ_i F_{\mu\nu}(x_i)\xi_i^\mu \xi_i^\nu \approx 0, \tag{1.5}$$

contain a *non-minimal term* connected with the *spin-magnetic field coupling in the rest frame.*

Since we know that we cannot separate positive from negative energies in the Dirac equation coupled to an external electromagnetic field with an exact Foldy-Wouthuysen transformation [5], we expect to have the same problems present also to the pseudo-classical level, where the pseudo-classical Foldy-Wouthuysen transformation is known [13] in absence of external electric fields [6]. Now in the present case this pseudo-classical transformation has to be reformulated by using Grassmann-valued electric charges: this further semiclassical approximation will allow to get an exact semi-classical separation of the positive and negative energies. Therefore, due to spin, differently from the case of scalar particles, already at the semi-classical level we must make a pseudo-classical Foldy-Wouthuysen transformation which puts the Dirac-like constraint at order Q_i in a form producing only *even* operators after quantization. In this way we shall identify the kind of semiclassical couplings of the spinning particle to an *external* electric field which are compatible with the separation of positive energies from negative ones. These couplings will appear in the semiclassical approximation of the solutions of the one-particle Dirac equation in all those cases in which an eikonal approximation plus a standard Foldy-Wouthuysen transformation will allow us to identify positive (or negative) energy spinning particle as a realistic approximation[7].

The next step will be to take the parametrized Minkowski action describing charged spinning particles and the electromagnetic field on arbitrary spacelike hypersurfaces, in which the minimal coupling describes correctly the magnetic couplings, and to *add non-minimally* these electric couplings of the spinning particles with the (*now dynamical*) electromagnetic field. Since we have not found the needed covariant modification of the Lagrangian, we limit ourselves to present the form of these couplings after the reduction to the Wigner hyperplanes, viz. in the rest-frame instant form of dynamics.

Therefore, we will have a modified invariant mass for the isolated system plus the ordinary rest-frame conditions, namely the vanishing of the total 3-momentum of the isolated system

---

[5] See Ref. [12] and its rich bibliography for the special cases in which this is possible and for the lack of a mathematical justification of the validity of the Foldy-Wouthuysen transformation.

[6] It produces forms of the Dirac-like constraint in which there are only polynomials in the Grassmann variables which after quantization become *even* operators.

[7] In particular the approximation will be valid as long as the electric fields are not too strong.

inside the Wigner hyperplane. This will produce a modification of the equations of motion. In particular the electromagnetic field equations will have a more complex particle source term, one dependent on the pseudoclassical spin variables. Therefore, the radiation gauge Lienard-Wiechert solutions of Ref. [9] will be modified.

At this point we have only to repeat the whole procedure, identified in the previous paper on charged scalar particles, to get the semi-classical phase space version of the *unique* semi-classical Lienard-Wiechert potential and electric field. Then we will make the canonical reduction to only particle degrees of freedom by imposing conditions on the electromagnetic field in the radiation gauge such that it coincide with this Hamiltonian Lienard-Wiechert electromagnetic field. After the identification of the new particle canonical variables, the final form of the Hamiltonian will identify the relativistic form of the *Darwin potential for spinning particles*.

In the 2-body case, the quantization of the lowest order in c^{-2} of the N=2 final Hamiltonian has to be compared with the positive-energy sector of the 3-dimensional positive-energy reduction of the *Bethe-Salpeter equation* [14–18] for 2-body bound states with spin 1/2 constituents of arbitrary mass (hydrogen atom, positronium, muonium). We find that our *Darwin*, *spin-orbit* and *spin-spin* terms reproduce *exactly* the standard Bethe-Salpeter result: this is the first time that, by making a relativistic separation of the center-of-mass motion and a subsequent study and canonical reduction of the relative motion, *results coming from quantum field theory can be exactly reproduced*.

The paper is organized as follows. In Section II we study the pseudo-classical Foldy-Wouthuysen transformation applied to the manifestly Lorentz-covariant spinning particle in an external electromagnetic field. A review of the theory of the positive-energy spinning particle interacting with a dynamical electromagnetic field in the rest-frame instant form of dynamics is given in Section III. In Section IV we introduce the non-minimal coupling of the electric field to the positive-energy spinning particles on the Wigner hyperplane using the results of Section II. In Section V we give the equations of motion for the particles and the field in the radiation gauge on the Wigner hyperplane. In Section VI we determine the unique semi-classical Lienard-Wiechert solution, we find its phase space expression, we add second class constraints to eliminate the radiation field, we find the Dirac brackets and the new canonical basis for the particles. The semi-classical Hamiltonian with the Darwin and spin-dependent potentials for the N-body problem is found in Section VII. In Section VIII we study the 2-body problem: we find the semiclassical Hamiltonian for muonium-, hydrogen- and positronium-like systems and then we quantize its lowest order part, reproducing the Bethe-Salpeter result. In the Conclusions after some comments we delineate how the research program could be developed to treat the non-Abelian case of the quark model.

In Appendix A there is the determination of some functions defined in Section II. In Appendix B, after a review on spacelike hypersurfaces, there is an attempt to extend the results of Section IV outside Wigner hyperplanes and outside the radiation gauge. In Appendix C there is the computation of the electromagnetic energy and 3-momentum when the Lienard-Wiechert solution is inserted in them. In Appendix D there the evaluation of some of the potentials of Section VII. In Appendix E there is the summation of the 2-body rest energy to a closed form.

II. THE PSEUDO-CLASSICAL FOLDY-WOUTHUYSEN TRANSFORMATION OF THE MANIFESTLY LORENTZ COVARIANT SPINNING PARTICLE.

In Ref. [11] there is the coupling of the manifestly Lorentz covariant spinning particle to external electromagnetic fields. It is based on the singular Lagrangian

$$
\begin{aligned}
L &= -\frac{i}{2}\xi_5\dot{\xi}_5 - \frac{i}{2}\xi_\mu\dot{\xi}^\mu + \\
&- \sqrt{m^2 - ieF_{\mu\nu}(x)\xi^\mu\xi^\nu}\sqrt{\Big(\dot{x}_\mu - \frac{i}{m}\xi_\mu\dot{\xi}_5\Big)^2} - e\dot{x}_\mu A^\mu(x) = \\
&= -\frac{i}{2}\xi_5\dot{\xi}_5 - \frac{i}{2}\xi_\mu\dot{\xi}^\mu - e\dot{x}_\mu A^\mu(x) - \\
&- \Big[m - \frac{ie}{2m}F_{\mu\nu}(x)\xi^\mu\xi^\nu - \frac{e^2}{m^3}F_{\mu\nu}(x)F_{\rho\lambda}(x)\xi^\mu\xi^\nu\xi^\rho\xi^\lambda\Big]\sqrt{\Big(\dot{x}_\mu - \frac{i}{m}\xi_\mu\dot{\xi}_5\Big)^2},
\end{aligned}
\tag{2.1}
$$

which, besides the standard minimal coupling, has a *non-minimal mass renormalization* $-ieF_{\mu\nu}\xi^\mu\xi^\nu = eF_{\mu\nu}S^{\mu\nu}$. The ie coefficient in front of $F_{\mu\nu}\xi^\mu\xi^\nu$, corresponding to the absence of an anomalous magnetic moment of the electron, is the only one ensuring that the two constraints remain first class even in presence of an external electromagnetic field.

Indeed, besides the second class constraints $\pi_\mu - \frac{i}{2}\xi_\mu \approx 0$ and the added one (like in the free case) $\pi_5 + \frac{i}{2}\xi_5 \approx 0$ [8], one gets the first class constraints

$$
\begin{aligned}
\chi_D &= (p_\mu - eA_\mu(x))\xi^\mu - m\xi_5 \approx 0\\
\chi &= (p - eA(x))^2 - m^2 + ieF_{\mu\nu}(x)\xi^\mu\xi^\nu \approx 0,
\end{aligned}
$$

$$
\{\chi_D, \chi_D\}^* = i\chi, \qquad \{\chi, \chi\}^* = \{\chi, \chi_D\}^* = 0. \tag{2.2}
$$

As said in the Introduction, if we have many particles we assume that the spin Grassmann variables of each particle commute with those of the other particles.

Following Ref. [19], we now describe also the electric charge of each particle in a semi-classical way by means of a pair of complex conjugate Grassmann variables [9] $\theta_i(\tau), \theta_i^*(\tau)$ [19] satisfying ($I_i = I_i^* = \theta_i^*\theta_i$ is the generator of the $U_{em}(1)$ group of particle i)

$$
\begin{aligned}
&\theta_i^2 = \theta_i^{*2} = 0, \qquad \theta_i\theta_i^* + \theta_i^*\theta_i = 0,\\
&\theta_i\theta_j = \theta_j\theta_i, \qquad \theta_i\theta_j^* = \theta_j^*\theta_i, \qquad \theta_i^*\theta_j^* = \theta_j^*\theta_i^*, \qquad i \neq j,
\end{aligned}
$$

$$
Q_i = e_i\theta_i^*\theta_i, \qquad Q_i^2 = 0, \qquad Q_iQ_j = Q_jQ_i. \tag{2.3}
$$

---

[8] After the elimination of the Grassmann momenta by going to Dirac brackets with respect to these second class constraints, the original Poisson brackets $\{x^\mu, p^\nu\} = \{\xi^\mu, \pi^\nu\} = -\eta^{\mu\nu}$, $\{\xi_5, \pi_5\} = -1$ become the following non-null Dirac bracket: $\{x^\mu, p^\nu\}^* = -\eta^{\mu\nu}$, $\{\xi^\mu, \xi^\nu\}^* = i\eta^{\mu\nu}$, $\{\xi_5, \xi_5\}^* = -i$; in what follows we will use the notation $\{.,.\}$ for these Dirac brackets.

[9] They are assumed to commute with the spin Grassmann variables.

The action now depends also on the configuration variables $\theta_i(\tau)$ and $\theta^*_i(\tau)$,$i = 1,..,N$, through an extra *kinetic* piece for the complex Grassmann charges $\int \frac{i}{2}[\theta^*_i(\tau)\dot{\theta}_i(\tau) - \dot{\theta}^*_i(\tau)\theta_i(\tau)]d\tau$,

The Grassmann momenta associated to these extra variables give rise to the second class constraints $\pi_{\theta\, i} + \frac{i}{2}\theta^*_i \approx 0$, $\pi_{\theta^*\, i} + \frac{i}{2}\theta_i \approx 0$ [10]; $\pi_{\theta\, i}$ and $\pi_{\theta^*\, i}$ are then eliminated with the help of Dirac brackets

$$\{A,B\}^* = \{A,B\} - i[\{A, \pi_{\theta\, i} + \frac{i}{2}\theta^*_i\}\{\pi_{\theta^*\, i} + \frac{i}{2}\theta_i, B\} + \{A, \pi_{\theta^*\, i} + \frac{i}{2}\theta_i\}\{\pi_{\theta\, i} + \frac{i}{2}\theta^*_i, B\}], \tag{2.4}$$

so that the remaining Grassmann variables have the fundamental Dirac brackets (which we will still denote $\{.,.\}$ for the sake of simplicity)

$$\begin{aligned} \{\theta_i(\tau), \theta_j(\tau)\} &= \{\theta^*_i(\tau), \theta^*_j(\tau)\} = 0, \\ \{\theta_i(\tau), \theta^*_j(\tau)\} &= -i\delta_{ij}. \end{aligned} \tag{2.5}$$

In Ref. [13] a pseudo-classical Foldy-Wouthuysen canonical transformation was introduced, which, after quantization, realizes an exact Foldy-Wouthuysen transformation of the Dirac equation in the case $A_0(x) = 0$, $\frac{\partial \vec{A}(x)}{\partial x^0} = 0$. However in that paper the electric charge was not Grassmann-valued.

The infinitesimal generator S of the pseudo-classical FW transformation of Ref. [13] is

$$S = 2i(\vec{\Pi} \cdot \vec{\xi})\xi_5\theta(\alpha), \tag{2.6}$$

where [in the free case we get $\alpha = \vec{p}^2$, $\theta(\alpha) = \theta(p)$]

$$\begin{aligned} p &= |\, \vec{p}\,|, \\ \vec{\Pi} &= \vec{p} - Q\vec{A}(x), \\ \alpha &= \vec{\Pi}^2 - iQF_{hk}(x)\xi^h\xi^k, \\ \sqrt{\alpha}\theta(\alpha) &= (1/2)\,\mathrm{arctg}\left(\frac{\sqrt{\alpha}}{m}\right). \end{aligned} \tag{2.7}$$

The action T_S of the canonical transformation generated by S on a generic function F in phase space is

$$T_S : F \to F + \{F, S\} + (1/2!)\{\{F,S\},S\} + ... \tag{2.8}$$

Let us now see what is the effect of the canonical transformation (2.8) when the electric charge Q is Grassmann-valued, so that we have $Q^2 = 0$. If we consider its action on the Dirac-like constraint

$$\chi_D = (p_0 - QA_0(x))\xi_0 - (\vec{p} - Q\vec{A}(x)) \cdot \vec{\xi} - m\xi_5 \approx 0, \tag{2.9}$$

[10] $\{\theta_i(\tau), \pi_{\theta j}(\tau)\} = \{\theta^*_i(\tau), \pi_{\theta^* j}(\tau)\} = -\delta_{ij}$; $\{\pi_{\theta\, i} + \frac{i}{2}\theta^*_i, \pi_{\theta^*\, j} + \frac{i}{2}\theta_j\} = -i\delta_{ij}$.

we get from Ref. [13] the result

$$T_S : (-\vec{\Pi}\cdot\vec{\xi} - m\xi_5) \to -\xi_5\sqrt{m^2+\alpha}. \tag{2.10}$$

We have now to evaluate the following two terms at order Q

$$T_S : -QA_0(x)\xi_0 \to -Q[A_0(x)\xi_0 + \{A_0(x)\xi_o, S\} + (1/2!)\{\{A_0(x)\xi_o, S\}, S\} + ...,$$

$$T_S : p_0\xi_0 \to p_0\xi_0 + \{p_0\xi_0, S\} + (1/2!)\{\{p_0\xi_0, S\}, S\} + ... \tag{2.11}$$

Using $\xi_5^2 = 0$, we can write:

$$\{\{...\{QA_0(x)\xi_0, \underbrace{S\}, S\}, ...\}, S\}}_{n} = \begin{cases} Q(2i)^n\Phi_n(p,x,\vec{\xi})\xi_5\xi_0 & n-odd \\ \\ Q(2i)^n\Phi_n(p,x,\vec{\xi})\xi_0 & n-even \end{cases} + \mathcal{O}(Q^2), \tag{2.12}$$

with the odd functions Φ_n defined in Appendix A. Dropping the Q^2 term we have:

$$T_S : -QA_0(x)\xi_0 \to -QF_1(p,x,\vec{\xi})\xi_0 - QG_1(p,x,\vec{\xi})\xi_5\xi_0, \tag{2.13}$$

with the following two functions (they are odd, i.e. linear in the three Grassmann variables $\vec{\xi}$)

$$\begin{cases} F_1(p,x,\vec{\xi}) &= \sum_{k=0}^{\infty}\frac{(2i)^{2k}}{2k!}\Phi_{2k}(p,x,\vec{\xi}), \\ \\ G_1(p,x,\vec{\xi}) &= \sum_{k=0}^{\infty}\frac{(2i)^{2k+1}}{(2k+1)!}\Phi_{2k+1}(p,x,\vec{\xi}). \end{cases} \tag{2.14}$$

Equally, we can write:

$$\{\{...\{p_0\xi_0, \underbrace{S\}, S\}, ...\}, S\}}_{n} = \begin{cases} Q(2i)^n\Psi_n(p,x,\vec{\xi})\xi_5\xi_0 & n-odd \\ \\ Q(2i)^n\Psi_n(p,x,\vec{\xi})\xi_0 & n-even, n\neq 0, \end{cases} \tag{2.15}$$

with the odd functions Ψ_n defined in Appendix A. Then we get

$$T_S : p_0(x)\xi_0 \to p_0\xi_0 + QF_2(p,x,\vec{\xi})\xi_0 + QG_2(p,x,\vec{\xi})\xi_5\xi_0, \tag{2.16}$$

with the following two functions (they are odd, i.e. linear, in the three Grassmann variables $\vec{\xi}$)

$$\begin{cases} F_2(p,x,\vec{\xi}) &= \sum_{k=1}^{\infty}\frac{(2i)^{2k}}{2k!}\Psi_{2k}(p,x,\vec{\xi}), \\ \\ G_2(p,x,\vec{\xi}) &= \sum_{k=0}^{\infty}\frac{(2i)^{2k+1}}{(2k+1)!}\Psi_{2k+1}(p,x,\vec{\xi}). \end{cases} \tag{2.17}$$

In total, we can sum the two term to get

$$T_S : (p_0 - QA_0(x))\xi_0 \to \Big[p_0 + QF(p,x,\vec{\xi})\Big]\xi_0 + QG(p,x,\vec{\xi})\xi_5\xi_0, \tag{2.18}$$

where $F = F_2 - F_1$ and $G = G_2 - G_1$. See Appendix A for the determination of the functions F and G. From Eqs. (A26), (A32) their final expression is $[\theta(p) = (1/2p)\,\mathrm{arctg}(p/m)]$

$$F(p,x,\vec{\xi}) = A_0(x) - i\frac{(\vec{p}\cdot\vec{\xi})\vec{\xi}\cdot\vec{E}(x)}{(m+\sqrt{m^2+\vec{p}^2})\sqrt{m^2+\vec{p}^2}},$$

$$\begin{aligned} G(p,x,\vec{\xi}) &= i\frac{sin\,[2p\theta(p)]}{p}\vec{\xi}\cdot\vec{\partial}A_o(x) + \\ &+ i\Big[2\sum_u \frac{\partial\theta(p)}{\partial p^u}\frac{\partial A_o(x)}{\partial x^u} - \frac{sin\,[2p\theta(p)] - 2p\theta(p)}{p^3}\vec{p}\cdot\vec{\partial}A_o(x)\Big]\vec{p}\cdot\vec{\xi}. \end{aligned} \tag{2.19}$$

The action of the canonical transformation T_S's on the Dirac-like constraint χ_D thus turns out to be

$$T_S : \chi_D \to \chi'_D = \Big[p_0 + QF(p,x,\vec{\xi})\Big]\xi_0 - \xi_5\sqrt{\alpha+m^2} + QG(p,x,\vec{\xi})\xi_5\xi_0. \tag{2.20}$$

The new constraint χ'_D contains some terms, linear in ξ_0 or ξ_5, corresponding to *even* diagonal terms in quantum case and some terms, linear in $\xi_5\xi_0$, corresponding to *odd* skew-diagonal terms, according to the rule: $\xi_5 \to \gamma_5, \xi_0 \to \gamma_5\gamma_0$:

$$\chi'_5 \to \gamma_5\left[\underbrace{(p_0+QF)\gamma_0 - \sqrt{m^2+\alpha}}_{diagonal} + \underbrace{QG\gamma_5\gamma_0}_{skew-diagonal}\right]. \tag{2.21}$$

In order to cancel the *odd* skew-diagonal terms at order Q, we define a unknown infinitesimal generator $W = Qw(p,x,\vec{\xi})$ for a new canonical transformation T_W

$$\begin{aligned} T_W : \chi'_D \to \chi''_D &= \chi'_D + Q\{\chi'_D, w\} = \\ &= \chi'_D + Q\{p_0\xi_0 - \xi_5\sqrt{m^2+\vec{p}^2}, w\}. \end{aligned} \tag{2.22}$$

By imposing the Ansatz:

$$w = w_1(p,x,\vec{\xi})\xi_0 + w_2(p,x,\vec{\xi})\xi_5, \tag{2.23}$$

the condition for the cancellation of the skew-diagonal terms is

$$\{p_0\xi_0 - \xi_5\sqrt{m^2+\vec{p}^2}, w_1(p,x,\vec{\xi})\xi_0 + w_2(p,x,\vec{\xi})\xi_5\} + G(p,x,\vec{\xi})\xi_5\xi_0 = 0. \tag{2.24}$$

This condition is equivalent to a partial differential equation for the unknown functions w_1, w_2:

$$(-i)w_1p_0 - (-i)w_2\sqrt{m^2+\vec{p}^2} - \{p_0, w_2\}\xi_5\xi_0 - \{\sqrt{m^2+\vec{p}^2}, w_1\}\xi_5\xi_0 + G\xi_5\xi_0 = 0,$$

$$\Rightarrow \begin{cases} w_1p_0 - w_2\sqrt{m^2+\vec{p}^2} = 0, \\ \\ -\{p_0, w_2\} - \{\sqrt{m^2+\vec{p}^2}, w_1\} + G = 0, \end{cases}$$

$$\Rightarrow \begin{cases} w_2 = \frac{p_0}{\sqrt{m^2+\vec{p}^2}}w_1, \\ \\ -\frac{\partial w_2}{\partial x^o} - \frac{1}{\sqrt{m^2+\vec{p}^2}}\vec{p}\cdot\partial w_1 + G = 0, \end{cases}$$

$$\Rightarrow -\frac{p_0}{\sqrt{m^2+\vec{p}^2}}\frac{\partial w_1}{\partial x^o} - \frac{1}{\sqrt{\vec{p}^2+m^2}}\vec{p}\cdot\vec{\partial}w_1 + G = 0. \tag{2.25}$$

By introducing the Green function $g_p(x)$ for $p^\mu \partial_\mu$ we find the following solution for w_1, w_2 [see Appendix B for the tetrads $\epsilon^\mu_A(u(p))$]

$$
\begin{aligned}
w_2 &= \frac{p_0}{\sqrt{m^2+\vec{p}^2}} w_1, \\
p^\mu \partial_\mu w_1(p,x,\vec{\xi}) &= \sqrt{m^2+\vec{p}^2} G(p,x,\vec{\xi}), \\
&\Downarrow \\
w_1(p,x,\vec{\xi}) &= \int d^4y\, g_p(x-y)\, \sqrt{m^2+\vec{p}^2} G(p,y,\vec{\xi}),
\end{aligned}
$$

$$
\begin{aligned}
& p_\mu \partial^\mu g_p(x) = \delta^4(x), \\
& g_p(x) = \pm \frac{\theta(\pm p^\mu x_\mu)\, \delta^3(\epsilon^\mu_r(u(p)) x_\mu)}{\sqrt{p^2}}.
\end{aligned}
\tag{2.26}
$$

This shows the existence of the canonical transformation T_W and the non-locality of the separation of positive and negative energies. Therefore, the *odd* term $QG\xi_5\xi_0$ of Eq.(2.20) can be eliminated at the order Q with the canonical transformation T_W.

After these two canonical transformations $T_W \circ T_S$, we get the following form of the Dirac-like constraint at order Q

$$
\begin{aligned}
\chi''_D &= \Big[p_0 - QA_0(x) + iQ(\vec{p}\cdot\xi)(\vec{\xi}\cdot\vec{E}(x)) \frac{1}{(m+\sqrt{m^2+\vec{p}^2})\sqrt{m^2+\vec{p}^2}}\Big]\xi_0 - \sqrt{m^2+\alpha}\,\xi_5 = \\
&= [p_0 - QA_0(x) - Q\frac{\vec{p}\cdot\vec{E}(x)\times\vec{S}}{(m+\sqrt{m^2+\vec{p}^2})\sqrt{m^2+\vec{p}^2}}\Big]\xi_0 - \sqrt{m^2+\alpha}\,\xi_5 \approx 0,
\end{aligned}
\tag{2.27}
$$

where we introduced the spin $\vec{S} = -\frac{i}{2}\vec{\xi}\times\vec{\xi}$.

To find the final form χ'' ($T_W \circ T_S : \chi \rightarrow \chi''$) of the mass-shell constraint we use the fact that $T_W \circ T_S$ is a canonical transformation, so that

$$
\{\chi''_D, \chi''_D\} = i\chi''.
\tag{2.28}
$$

We get ($F_{hk} = \epsilon_{hkr} B_r$)

$$
\begin{aligned}
\chi'' = &\Big(p_0 - QA_0(x) - Q\vec{p}\cdot\vec{E}(x)\times\vec{S}\frac{m-\sqrt{m^2+\vec{p}^2}}{\vec{p}^2\sqrt{m^2+\vec{p}^2}}\Big)^2 - \\
&- \Big[m^2 + (\vec{p}-Q\vec{A}(x))^2 + 2Q\vec{S}\cdot\vec{B}(x)\Big] \approx 0,
\end{aligned}
\tag{2.29}
$$

after having discarded terms proportional to $\xi_0\xi_5$, since they vanish due to $\chi''_D \approx 0$, which implies ξ_5 weakly proportional to ξ_o at order Q.

The mass-shell constraint can to be resolved at order Q in two constraints, corresponding to the two signs of the energy:

$$
\begin{aligned}
\chi^{(\pm)} &= p_0 - QA_0(x) + iQ(\vec{p}\cdot\xi)(\vec{\xi}\cdot\vec{E}(x))\frac{1}{(m+\sqrt{m^2+\vec{p}^2})\sqrt{m^2+\vec{p}^2}} \mp \\
&\mp \sqrt{m^2+(\vec{p}-Q\vec{A}(x))^2 - iQ\xi^h\xi^k F_{hk}(x)} = \\
&= p_o - QA_o(x) - \frac{Q\vec{p}\cdot\vec{E}(x)\times\vec{S}}{(m+\sqrt{m^2+\vec{p}^2})\sqrt{m^2+\vec{p}^2}} \mp \\
&\mp \sqrt{m^2+(\vec{p}-Q\vec{A}(x))^2 + 2Q\vec{S}\cdot\vec{B}(x)} \approx 0.
\end{aligned}
\tag{2.30}
$$

In this way we have identified the extra coupling at order Q to the electric field $\vec{E}(x) = -\frac{\partial \vec{A}(x)}{\partial x^0} - \vec{\partial} A_0(x)$ of a spinning particle with a definite sign of the energy. *Only* the semi-classical approximation $Q^2 = 0$ allows us to get this result in closed form.

Now the problem is to interpret this result. Since $A_0(x)$ and $\vec{A}(x)$ describe an *external* electromagnetic field, we are in an arbitrary fixed gauge. But we will need the result in the radiation gauge with both transverse radiation fields and action-at-a-distance Coulomb potentials from other charges acting simultaneously on the given charged spinning particle. It seems reasonable to interpret $A_0(x)$ as the scalar Coulomb potential generated by the other charges. Then the term $-\frac{\partial \vec{A}(x)}{\partial x^0} - \vec{\partial} A_0(x)$ should be interpreted as the *sum of the transverse radiation electric field and of the gradient of the Coulomb potential* (action-at-a-distance electric field generated by the other charges), since this is the total electric field acting on the charged spinning particle.

III. THE POSITIVE-ENERGY SPINNING PARTICLE.

In this Section we shall review the description of the positive-energy charged spinning particle given in Ref. [9]. To define a charged spinning particle with a definite sign of the energy on spacelike hypersurfaces[11], the starting point was the Lagrangian description of a charged scalar particle (see Refs. [4,20]) with only a real Grassmann 4-vector $\xi^\mu(\tau)$ for the description of spin[12]. After the Legendre transformation to the Hamiltonian formalism, a Hamiltonian odd second class constraint [13] is added to eliminate one of the components of ξ^μ. In this way only three Grassmann variables will survive for each particle and, after quantization, they will generate Pauli matrices acting on the 2-spinors describing the positive-energy wave functions. It turns out that the addition by hand of this constraint gives a consistent set of constraints.

Since, due to this last constraint, the Lagrangian description is too complicated [14] the model was defined directly in phase space by means of a set of constraints. As usual in relativistic particle mechanics, only the Hamiltonian description is tractable, because the Lagrangian one is too involved and very often it is impossible to get it in closed form.

Given a 3+1 splitting of Minkowski spacetime with a foliation whose spacelike leaves are defined by the embeddings $\Sigma \mapsto \Sigma_\tau$, $(\tau, \vec{\sigma}) \mapsto z^\mu(\tau, \vec{\sigma})$, the embeddings become new configuration variables describing all possible hypersurfaces (all possible congruences of timelike accelerated observers). Since, due to the separate τ- and $\vec{\sigma}$-reprametrization invariances of the action, there are first class constraints implying the independence of the description from the choice of the 3+1 splitting, the $z^\mu(\tau, \vec{\sigma})$'s are the *gauge* variables of this type of general covariance[15] See Appendix B for the definition of the notations and of the induced metric. Each positive energy particle is described by the coordinates $\vec{\eta}_i(\tau)$ such that

---

[11]See Appendix B for some notions on spacelike hypersurfaces.

[12]To describe a positive-energy spinning particle we have to solve the first class constraint $\chi_D \approx 0$ of Eq.(1.3) to express ξ_5 in terms of the Grassmann 4-vector ξ^μ. The needed gauge fixing to $\chi_D \approx 0$ is a constraint eliminating one of the four components of ξ^μ: as shown in Ref. [9] in the free case it is $p_\mu \xi^\mu + m\xi^5 \approx 0$, so that $\xi_5 \approx 0$ and $p_\mu \xi^\mu \approx 0$ hold simultaneously. The mass-shell constraint $\chi \approx 0$ of Eq.(1.1) is eliminated by the choice of the energy like for scalar particles.

[13]It is of the transversality type $p_\mu \xi^\mu \approx 0$, but with p_μ being the conserved total momentum of the isolated system and *not* the particle momentum.

[14]In Eq.(29) of Ref. [9] there is the Lagrangian generating all the constraints (3.3) except the transversality ones $\phi_i(\tau) = p_\mu \xi^\mu_i \approx 0$. The Lagrangian generating all the constraints (3.3) could be recovered by inverse Legendre transformation. This has been done in Ref. [9] [see its Eq.(50)] only in absence of electromagnetic field, because the general case is very complicated and not particularly interesting, except for the determination of the energy-momentum tensor. In any case, the Lagrangian contains both a minimal coupling of the positive-energy particles to the electromagnetic field and a non-minimal coupling like in Eq.(2.1).

[15]The descriptions given by arbitrary congruences of timelike observers are gauge equivalent.

$x_i^\mu(\tau) = z^\mu(\tau, \vec{\eta}_i(\tau))$. Moreover, for each particle there will be a Grassmann 4-vector $\xi_i^\mu(\tau)$ for the spin and a pair $\theta_i(\tau)$, $\theta_i^*(\tau)$ of complex scalar Grassmann variables for the electric charge.

On the hypersurface Σ_τ, we describe the electromagnetic potential and field strength with Lorentz-scalar variables $A_{\check{A}}(\tau, \vec{\sigma})$ and $F_{\check{A}\check{B}}(\tau, \vec{\sigma})$ respectively, which have the *equal time* concept introduced by the embedding, defined by [$\tilde{A}$, $\tilde{F}_{\mu\nu}$ are the standard electromagnetic potentials and filed strengths, respectively]

$$
\begin{aligned}
&A_{\check{A}}(\tau, \vec{\sigma}) = z^\mu_{\check{A}}(\tau, \vec{\sigma}) \tilde{A}_\mu(z(\tau, \vec{\sigma})),\\
&F_{\check{A}\check{B}}(\tau, \vec{\sigma}) = \partial_{\check{A}} A_{\check{B}}(\tau, \vec{\sigma}) - \partial_{\check{B}} A_{\check{A}}(\tau, \vec{\sigma}) = z^\mu_{\check{A}}(\tau, \vec{\sigma}) z^\nu_{\check{B}}(\tau, \vec{\sigma}) \tilde{F}_{\mu\nu}(z(\tau, \vec{\sigma})).
\end{aligned} \tag{3.1}
$$

The model of Ref. [9] is defined in a phase space spanned by the canonically conjugate pairs of variables $z^\mu(\tau, \vec{\sigma})$, $\rho_\mu(\tau, \vec{\sigma})$; $A_{\check{A}}(\tau, \vec{\sigma})$, $\pi^{\check{A}}(\tau, \vec{\sigma})$ [16]; $\vec{\eta}_i(\tau)$, $\vec{\kappa}_i(\tau)$; $\xi_i^\mu(\tau)$, $\pi_i^\mu(\tau)$; $\theta_i(\tau)$, $\pi_{\theta i}(\tau)$; $\theta_i^*(\tau)$, $\pi_{\theta^* i}(\tau)$ with the following Poisson brackets

$$
\begin{aligned}
&\{z^\mu(\tau, \vec{\sigma}), \rho_\nu(\tau, \vec{\sigma}^{'}\} = -\eta^\mu_\nu \delta^3(\vec{\sigma} - \vec{\sigma}^{'}),\\
&\{A_{\check{A}}(\tau, \vec{\sigma}), \pi^{\check{B}}(\tau, \vec{\sigma}^{'})\} = \eta^{\check{B}}_{\check{A}} \delta^3(\vec{\sigma} - \vec{\sigma}^{'}),\\
&\{\eta_i^{\check{r}}(\tau), \kappa_{j\check{s}}(\tau)\} = -\delta_{ij}\delta^{\check{r}}_{\check{s}},\\
&\{\theta_i(\tau), \pi_{\theta\, j}(\tau)\} = -\delta_{ij},\\
&\{\theta^*_i(\tau), \pi_{\theta^*\, j}(\tau)\} = -\delta_{ij},\\
&\{\xi_i^\mu, \pi_j^\nu\} = -\delta_{ij}\eta^{\mu\nu}.
\end{aligned} \tag{3.2}
$$

The total conserved 4-momentum of the system is $p_s^\mu = \int d^3\sigma \rho^\mu(\tau, \vec{\sigma})$ [see Eqs.(3.7)].

The model is defined by the following set of constraints (only positive energy particles are considered)

$$
\begin{aligned}
&\pi_{\theta\, i} + \frac{i}{2}\theta_i^* \approx 0,\\
&\pi_{\theta^*\, i} + \frac{i}{2}\theta_i \approx 0,
\end{aligned}
$$

$$
\begin{aligned}
&\phi_i(\tau) = [\pi_i^\mu(\tau) + \frac{i}{2}\xi_i^\mu(\tau)] \int d^3\sigma\, \rho_\mu(\tau, \vec{\sigma}) = [\pi_i^\mu(\tau) + \frac{i}{2}\xi_i^\mu(\tau)] p_{s\mu} \approx 0,\\
&\chi_i^\mu(\tau) = \pi_i^\mu(\tau) - \frac{i}{2}\xi_i^\mu(\tau) \approx 0, \quad \Rightarrow \phi_i(\tau) \approx p_{s\mu}\xi_i^\mu(\tau) \approx 0,
\end{aligned}
$$

$$
\begin{aligned}
&\pi^\tau(\tau, \vec{\sigma}) \approx 0\\
&\Gamma(\tau, \vec{\sigma}) = \partial_{\check{r}} \pi^{\check{r}}(\tau, \vec{\sigma}) - \sum_{i=1}^N Q_i(\tau) \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) \approx 0,\\
&\mathcal{H}_\mu(\tau, \vec{\sigma}) = \rho_\mu(\tau, \vec{\sigma}) - l_\mu(\tau, \vec{\sigma}) \Big[- \frac{1}{2\sqrt{\gamma(\tau, \vec{\sigma})}} \pi^{\check{r}}(\tau, \vec{\sigma}) g_{\check{r}\check{s}}(\tau, \vec{\sigma}) \pi^s(\tau, \vec{\sigma}) +
\end{aligned}
$$

---

[16] $E_{\check{r}} = F_{\check{r}\tau}$ and $B_{\check{r}} = \frac{1}{2}\epsilon_{\check{r}\check{s}\check{t}} F_{\check{s}\check{t}}$ ($\epsilon_{\check{r}\check{s}\check{t}} = \epsilon^{\check{r}\check{s}\check{t}}$) are the electric and magnetic fields respectively; for $g_{\check{A}\check{B}} \to \eta_{\check{A}\check{B}}$ one gets $\pi^{\check{r}} = -E_{\check{r}} = E^{\check{r}}$.

$$
\begin{aligned}
&+ \frac{\sqrt{\gamma(\tau,\vec{\sigma})}}{4}\gamma^{\check{r}\check{s}}(\tau,\vec{\sigma})\gamma^{\check{u}\check{v}}(\tau,\vec{\sigma})F_{\check{r}\check{u}}(\tau,\vec{\sigma})F_{\check{s}\check{v}}(\tau,\vec{\sigma}) + \\
&+ \sum_{i=1}^{N}\delta^3(\vec{\sigma}-\vec{\eta}_i(\tau))\cdot \\
&\cdot\ \sqrt{m_i^2 - \gamma^{\check{r}\check{s}}(\tau,\vec{\sigma})(\kappa_{i\check{r}}(\tau)-Q_i(\tau)A_{\check{r}}(\tau,\vec{\sigma}))(\kappa_{i\check{s}}(\tau)-Q_i(\tau)A_{\check{s}}(\tau,\vec{\sigma}))} + \\
&+ \frac{1}{2\sqrt{\gamma(\tau,\vec{\sigma})}}\sum_{i,j=1}^{N}\delta^3(\vec{\sigma}-\vec{\eta}_i(\tau))\delta^3(\vec{\sigma}-\vec{\eta}_j(\tau))\cdot \\
&\cdot\ \frac{Q_i(\tau)Q_j(\tau)\xi_i^{\gamma}(\tau)\xi_i^{\delta}(\tau)l_{\gamma}(\tau,\vec{\sigma})\eta_{\delta\beta}\xi_j^{\alpha}(\tau)\xi_j^{\beta}(\tau)l_{\alpha}(\tau,\vec{\sigma})}{\sqrt{m_i^2-\gamma^{\check{r}\check{s}}(\tau,\vec{\sigma})\kappa_{i\check{r}}(\tau)\kappa_{i\check{s}}(\tau)}\sqrt{m_j^2-\gamma^{\check{r}\check{s}}(\tau,\vec{\sigma})\kappa_{j\check{r}}(\tau)\kappa_{j\check{s}}(\tau)}} + \\
&+ \frac{i}{\sqrt{\gamma(\tau,\vec{\sigma})}}\sum_{i=1}^{N}\delta^3(\vec{\sigma}-\vec{\eta}_i(\tau))\cdot \\
&\cdot\ \frac{Q_i(\tau)\xi_i^{\alpha}(\tau)\xi_i^{\beta}(\tau)}{\sqrt{m_i^2-\gamma^{\check{r}\check{s}}(\tau,\vec{\sigma})\kappa_{i\check{r}}(\tau)\kappa_{i\check{s}}(\tau)}}l_{\alpha}(\tau,\vec{\sigma})z_{\check{s}\beta}(\tau,\vec{\sigma})\pi^{\check{s}}(\tau,\vec{\sigma}) - \\
&- \frac{i}{2}\sum_{i=1}^{N}\delta^3(\vec{\sigma}-\vec{\eta}_i(\tau))\frac{Q_i(\tau)\xi_i^{\alpha}(\tau)\xi_i^{\beta}(\tau)}{\sqrt{m_i^2-\gamma^{\check{r}\check{s}}(\tau,\vec{\sigma})\kappa_{i\check{r}}(\tau)\kappa_{i\check{s}}(\tau)}}\cdot \\
&\cdot\ z_{\check{u}\alpha}(\tau,\vec{\sigma})z_{\check{v}\beta}(\tau,\vec{\sigma})\gamma^{\check{r}\check{u}}(\tau,\vec{\sigma})\gamma^{\check{s}\check{v}}(\tau,\vec{\sigma})F_{\check{r}\check{s}}(\tau,\vec{\sigma})\Big] - \\
&\\
&- \gamma^{\check{r}\check{s}}(\tau,\vec{\sigma})z_{\check{s}\mu}(\tau,\vec{\sigma})\Big[F_{\check{r}\check{u}}(\tau,\vec{\sigma})\pi^{\check{u}}(\tau,\vec{\sigma}) + \\
&+ \sum_{i=1}^{N}\delta^3(\vec{\sigma}-\vec{\eta}_i(\tau))\Big(\kappa_{\check{r}i}-Q_i(\tau)A_{\check{r}}(\tau,\vec{\sigma})\Big)\Big] \approx 0.
\end{aligned}
\tag{3.3}
$$

The Lagrangian density that generates all these constraints except for the combination $\phi_i(\tau)=\int d^3\sigma\,\rho_\mu(\tau,\vec{\sigma})\xi_i^\mu(\tau)\approx 0$ of the third and fourth set above is given by [see Eq.(29) of Ref. [9]]

$$
\begin{aligned}
\mathcal{L}\,(\tau,\vec{\sigma}) &= \sum_{i=1}^{N}\delta^3(\vec{\sigma}-\vec{\eta}_i(\tau))\Big\{\frac{i}{2}(\theta_i^*(\tau)\dot{\theta}_i(\tau)-\dot{\theta}_i^*(\tau)\theta_i(\tau)) - \frac{i}{2}\xi_{\mu i}(\tau)\dot{\xi}_i^{\mu}(\tau) - \\
&- \sqrt{m_i^2 - iQ_i(\tau)\xi_i^{\mu}(\tau)\xi_i^{\nu}(\tau)z_{\mu}^{\check{A}}(\tau,\vec{\sigma})z_{\nu}^{\check{B}}(\tau,\vec{\sigma})F_{\check{A}\check{B}}(\tau,\vec{\sigma})}\cdot \\
&\sqrt{g_{\tau\tau}(\tau,\vec{\sigma})+2g_{\tau\check{r}}(\tau,\vec{\sigma})\dot{\eta}_i^{\check{r}}(\tau)+g_{\check{r}\check{s}}(\tau,\vec{\sigma})\dot{\eta}_i^{\check{r}}(\tau)\dot{\eta}_i^{\check{s}}(\tau)} - \\
&- Q_i(\tau)\Big(A_{\tau}(\tau,\vec{\sigma})+\dot{\eta}_i^{\check{r}}A_{\check{r}}(\tau,\vec{\sigma})\Big)\Big\} - \frac{\sqrt{g(\tau,\vec{\sigma})}}{4}F_{\check{A}\check{B}}(\tau,\vec{\sigma})F_{\check{C}\check{D}}(\tau,\vec{\sigma})g^{\check{A}\check{C}}(\tau,\vec{\sigma})g^{\check{B}\check{D}}(\tau,\vec{\sigma}).
\end{aligned}
\tag{3.4}
$$

The following Dirac Hamiltonian (see Eqs.(46) and (47) of Ref. [9] for its determination) completes the definition of our model

$$
H_D^F = \int d^3\sigma\Big[\lambda^{\mu}(\tau,\vec{\sigma})\mathcal{H}_{\mu}^*(\tau,\vec{\sigma}) - A_{\tau}(\tau,\vec{\sigma})\Gamma(\tau,\vec{\sigma}) + \mu_{\tau}(\tau,\vec{\sigma})\pi^{\tau}(\tau,\vec{\sigma})\Big],
$$

$$\mathcal{H}^*_\mu(\tau,\vec{\sigma}) = \mathcal{H}_\mu(\tau,\vec{\sigma}) + i\sum_{i=1}^N \{\mathcal{H}_\mu(\tau,\vec{\sigma}), \chi_i^\nu(\tau)\}\chi_{i\nu}(\tau) +$$
$$-\frac{i}{p_s^2}\sum_{i=1}^N \{\mathcal{H}_\mu(\tau,\vec{\sigma}), \phi_i(\tau)\}\phi_i(\tau) \approx 0. \qquad (3.5)$$

See Ref. [9] for their Poisson brackets. The constraints $\pi_{\theta\, i} + \frac{i}{2}\theta_i^* \approx 0$, $\pi_{\theta^*\, i} + \frac{i}{2}\theta_i \approx 0$, $\phi_i \approx 0$, $\chi_i^\mu \approx 0$ are second class, while the other are first class.

The Dirac multipliers $\lambda^\mu(\tau,\vec{\sigma})$, $\mu_\tau(\tau,\vec{\sigma})$ in the Dirac Hamiltonian imply that the constraints $\mathcal{H}^*_\mu(\tau,\vec{\sigma}) \approx 0$, $\pi^\tau(\tau,\vec{\sigma}) \approx 0$ are primary constraints of the unknown Lagrangian; the other primary constraints are the second class ones, whose associated Dirac multipliers are determined by the Dirac algorithm as shown in Ref. [9]. The Gauss law $\Gamma(\tau,\vec{\sigma}) \approx 0$ is the only secondary constraints and it appears in the Dirac Hamiltonian with $A_\tau(\tau,\vec{\sigma})$ as a multiplier not determined by the theory (it is a gauge variable).

As said in Section II the Grassmann momenta $\pi_{\theta i}(\tau)$, $\pi_{\theta^* i}(\tau)$ may be eliminated by using the Dirac brackets (2.5).

The conserved Poincaré generators are

$$p^\mu_s = \int d^3\sigma \rho^\mu(\tau,\vec{\sigma})$$
$$J^{\mu\nu} = \int d^3\sigma \Big[z^\mu(\tau,\vec{\sigma})\rho^\nu(\tau,\vec{\sigma}) - z^\nu(\tau,\vec{\sigma})\rho^\mu(\tau,\vec{\sigma})\Big] - \sum_{i=1}^N \Big[\xi_i^\mu(\tau)\pi_i^\nu(\tau) - \xi_i^\nu(\tau)\pi_i^\mu(\tau)\Big]. \qquad (3.6)$$

Since p^μ_s is a constant of the motion independent of the isolated system under investigation, we may write $\phi_i = (\pi_i^\mu + \frac{i}{2}\xi_i^\mu)p_{s\mu} \approx 0$. Due to the transversality to the total conserved 4-momentum[17], we have the possibility of reducing the ξ_i^μ's from 4 to 3 for each particle independently from the interactions.

As we see, the component of $\mathcal{H}_\mu(\tau,\vec{\sigma})$ along $l^\mu(\tau,\vec{\sigma})$ (i.e. orthogonal to Σ_τ) contains the electromagnetic energy density and also *spin-spin*, *spin-electric field* and *spin-magnetic field* interactions[18]. All of them are necessary to get the first class property for these constraints. Instead the components of $\mathcal{H}_\mu(\tau,\vec{\sigma})$ along $z^\mu_{\check{r}}(\tau,\vec{\sigma})$)(i.e. tangent to Σ_τ) contain only the electromagnetic Poynting vector as with scalar particles [4].

A. The restriction to Wigner's hyperplanes

As shown in Ref. [4], the restriction from arbitrary hypersurfaces Σ_τ to hyperplanes $\Sigma_{H\tau}$ is done by introducing the gauge-fixings by pairing up the first class constraints $\mathcal{H}^*_\nu(\tau,\vec{\sigma}')$ with

---

[17] This is a highly non-local property, since the whole hypersurface Σ_τ is involved in the reduction. Note that p^μ_s weakly coincides with the total 4-momentum of the isolated system by using $\mathcal{H}_\mu(\tau,\vec{\sigma}) \approx 0$.

[18] Only the last one will survive in the rest frame.

$$\zeta^\mu(\tau,\vec{\sigma}) = z^\mu(\tau,\vec{\sigma}) - x^\mu_s(\tau) - b^\mu_{\check{r}}(\tau)\sigma^{\check{r}} \approx 0,$$

$$\{\zeta^\mu(\tau,\vec{\sigma}), \mathcal{H}^*_\nu(\tau,\vec{\sigma}')\} = -\eta^\mu_\nu \delta^3(\vec{\sigma}-\vec{\sigma}'), \tag{3.7}$$

and the Dirac brackets which eliminate these variables are

$$\{A,B\}^* = \{A,B\} - \int d^3\sigma \{A, \zeta^\mu(\tau,\vec{\sigma})\}\{\mathcal{H}^*_\mu(\tau,\vec{\sigma}), B\} + \int d^3\sigma \{A, \mathcal{H}^*_\mu(\tau,\vec{\sigma})\}\{\zeta^\mu(\tau,\vec{\sigma}), B\}. \tag{3.8}$$

The hyperplane $\Sigma_{H\tau}$ is now described by just 10 configuration variables: an origin $x^\mu_s(\tau)$ and the 6 independent degrees of freedom in an orthonormal tetrad $b^\mu_{\check{A}}(\tau)$ $[b^\mu_{\check{A}}\,\eta_{\mu\nu} b^\nu_{\check{B}} = \eta_{\check{A}\check{B}}]$ with $b^\mu_\tau = l^\mu$, where l^μ is the τ-independent normal to the hyperplane. We have $z^\mu_{\check{r}}(\tau,\vec{\sigma}) \equiv b^\mu_{\check{r}}(\tau)$, $z^\mu_\tau(\tau,\vec{\sigma}) \equiv \dot{x}^\mu_s(\tau) + b^\mu_{\check{r}}(\tau)\sigma^{\check{r}}$, $g_{\check{r}\check{s}}(\tau,\vec{\sigma}) \equiv -\delta_{\check{r}\check{s}}$, $\gamma^{\check{r}\check{s}}(\tau,\vec{\sigma}) \equiv -\delta^{\check{r}\check{s}}$, $\gamma(\tau,\vec{\sigma}) = det\, g_{\check{r}\check{s}}(\tau,\vec{\sigma}) \equiv 1$. The non-vanishing Dirac brackets of the variables x^μ_s, p^μ_s, $b^\mu_{\check{A}}$, $S^{\mu\nu}_s$, $A_{\check{A}}$, $\pi^{\check{A}}$, ξ^μ_i, π^ν_j are

$$\begin{aligned}
\{x^\mu_s(\tau), p^\nu_s\}^* &= -\eta^{\mu\nu},\\
\{\eta^{\check{r}}_i(\tau), \kappa^{\check{s}}_j(\tau)\}^* &= \delta_{ij}\delta^{\check{r}\check{s}},\\
\{S^{\mu\nu}_s(\tau), b^\rho_{\check{A}}\}^* &= \eta^{\rho\nu} b^\mu_{\check{A}}(\tau) - \eta^{\rho\mu} b^\nu_{\check{A}}(\tau),\\
\{S^{\mu\nu}_s(\tau), S^{\alpha\beta}_s(\tau)\}^* &= C^{\mu\nu\alpha\beta}_{\gamma\delta} S^{\gamma\delta}_s(\tau),\\
\{\xi^\mu_i(\tau), \pi^\nu_j(\tau)\}^* &= -\eta^{\mu\nu}\delta_{ij}.
\end{aligned} \tag{3.9}$$

While p^μ_s is the momentum conjugate to x^μ_s, the 6 independent momenta conjugate to the 6 degrees of freedom in the $b^\mu_{\check{A}}$'s are hidden in $S^{\mu\nu}_s$, which is a component of the angular momentum tensor

$$\begin{aligned}
J^{\mu\nu} &= L^{\mu\nu}_s + S^{\mu\nu}_s + S^{\mu\nu}_\xi,\\
L^{\mu\nu}_s &= x^\mu_s(\tau) p^\nu_s - x^\nu_s(\tau) p^\mu_s,\\
S^{\mu\nu}_s &= b^\mu_{\check{r}}(\tau) \int d^3\sigma \sigma^{\check{r}} \rho^\nu(\tau,\vec{\sigma}) - b^\nu_{\check{r}}(\tau) \int d^3\sigma \sigma^{\check{r}} \rho^\mu(\tau,\vec{\sigma}),\\
S^{\mu\nu}_\xi &= -\sum_{i=1}^N \Big(\xi^\mu_i(\tau)\pi^\nu_i(\tau) - \xi^\nu_i(\tau)\pi^\mu_i(\tau)\Big),
\end{aligned}$$

$$\begin{aligned}
&\{J^{\mu\nu}, J^{\alpha\beta}\}^* = C^{\mu\nu\alpha\beta}_{\gamma\delta} J^{\gamma\delta}, \qquad \{L^{\mu\nu}_s, L^{\alpha\beta}_s\}^* = C^{\mu\nu\alpha\beta}_{\gamma\delta} L^{\gamma\delta}_s,\\
&\{S^{\mu\nu}_s, S^{\alpha\beta}_s\}^* = C^{\mu\nu\alpha\beta}_{\gamma\delta} S^{\gamma\delta}_s, \qquad \{S^{\mu\nu}_\xi, S^{\alpha\beta}_\xi\}^* = C^{\mu\nu\alpha\beta}_{\gamma\delta} S^{\gamma\delta}_\xi,\\
&C^{\mu\nu\alpha\beta}_{\gamma\delta} = \eta^\nu_\gamma \eta^\alpha_\delta \eta^{\mu\beta} + \eta^\mu_\gamma \eta^\beta_\delta \eta^{\nu\alpha} - \eta^\nu_\gamma \eta^\beta_\delta \eta^{\mu\alpha} - \eta^\mu_\gamma \eta^\alpha_\delta \eta^{\nu\beta}.
\end{aligned} \tag{3.10}$$

Next, we eliminate the second class constraints $\chi^\mu_i(\tau) \approx 0$, $\phi_i(\tau) \approx 0$ with the new Dirac brackets

$$\{A,B\}^*_D = \{A,B\}^* + i\sum_{i=1}^N \{A, \chi^\mu_i(\tau)\}^* \eta_{\mu\nu} \{\chi^\nu_i(\tau), B\}^* - \frac{i}{p^2_s} \sum_{i=1}^N \{A, \phi_i(\tau)\}^* \{\phi_i(\tau), B\}^*. \tag{3.11}$$

Now we have

$$
\begin{aligned}
&\pi^\mu_i(\tau) \equiv \frac{i}{2}\xi^\mu_i(\tau),\\
&\xi^\mu_i(\tau) p_{s\mu} \equiv 0,\\
\Rightarrow\ &\xi^\mu_i(\tau) \equiv \xi^\mu_{i\perp}(\tau) \equiv \Pi^{\mu\nu}\xi_{i\nu}(\tau) = \Big(\eta^{\mu\nu} - \frac{p^\mu_s p^\nu_s}{p^2_s}\Big)\xi_{i\nu}(\tau),\\
&S^{\mu\nu}_\xi \equiv -i\sum_{i=1}^N \xi^\mu_i \xi^\nu_i, \qquad p_{s\mu} S^{\mu\nu}_\xi \equiv 0.
\end{aligned}
\tag{3.12}
$$

However, now we get the following non-canonical Dirac brackets on $\Sigma_{H\tau}$

$$
\begin{aligned}
&\{x^\mu_s(\tau), x^\nu_s(\tau)\}^*_D = -i\sum_{i=1}^N \frac{\xi^\mu_i(\tau)\xi^\nu_i(\tau)}{p^2_s} \equiv \frac{S^{\mu\nu}_\xi(\tau)}{p^2_s},\\
&\{x^\mu_s(\tau), \xi^\nu_i(\tau)\}^*_D = \frac{\xi^\mu_i(\tau) p^\nu_s}{p^2_s},\\
&\{\xi^\mu_i(\tau), \xi^\nu_j(\tau)\}^*_D = i\Big(\eta^{\mu\nu} - \frac{p^\mu_s p^\nu_s}{p^2_s}\Big)\delta_{ij} \equiv i\Pi^{\mu\nu}\delta_{ij}.
\end{aligned}
\tag{3.13}
$$

In this way, we have eliminated the components of ξ^μ_i parallel to p^μ_s in a Lorentz-invariant way. The spin of each particle is described only by 3 Grassmann variables and the spin tensor $S^{\mu\nu}_\xi$ satisfies a Weyssenhoff condition. The angular momentum tensor becomes

$$
\begin{aligned}
&J^{\mu\nu} = L^{\mu\nu}_s + S^{\mu\nu}_s + S^{\mu\nu}_\xi,\\
&S^{\mu\nu}_\xi = -i\sum_{i=1}^N \xi^\mu_i(\tau)\xi^\nu_i(\tau),\\
&\{J^{\mu\nu}, J^{\alpha\beta}\}^*_D = \{J^{\mu\nu}, J^{\alpha\beta}\}^* = C^{\mu\nu\alpha\beta}_{\gamma\delta} J^{\gamma\delta},\\
&\{L^{\mu\nu}_s, L^{\alpha\beta}_s\}^*_D = C^{\mu\nu\alpha\beta}_{\gamma\delta} L^{\gamma\delta}_s - P^{\mu\nu\alpha\beta}_{\gamma\delta} S^{\gamma\delta}_\xi,\\
&\{S^{\mu\nu}_\xi, S^{\alpha\beta}_\xi\}^*_D = C^{\mu\nu\alpha\beta}_{\gamma\delta} S^{\gamma\delta}_\xi - P^{\mu\nu\alpha\beta}_{\gamma\delta} S^{\gamma\delta}_\xi,\\
&\{L^{\mu\nu}_s, S^{\alpha\beta}_\xi\}^*_D = P^{\mu\nu\alpha\beta}_{\gamma\delta} S^{\gamma\delta}_\xi,\\
&\{L^{\mu\nu}_s + S^{\mu\nu}_\xi, L^{\alpha\beta}_s + S^{\alpha\beta}_\xi\}^*_D = C^{\mu\nu\alpha\beta}_{\gamma\delta}\Big(L^{\gamma\delta}_s + S^{\gamma\delta}_\xi\Big),\\
&P^{\mu\nu\alpha\beta}_{\gamma\delta} \equiv \frac{p^\mu_s p^\beta_s}{p^2_s}\eta^\nu_\gamma\eta^\alpha_\delta + \frac{p^\nu_s p^\alpha_s}{p^2_s}\eta^\mu_\gamma\eta^\beta_\delta - \frac{p^\mu_s p^\alpha_s}{p^2_s}\eta^\nu_\gamma\eta^\beta_\delta - \frac{p^\nu_s p^\beta_s}{p^2_s}\eta^\mu_\gamma\eta^\alpha_\delta.
\end{aligned}
\tag{3.14}
$$

Since by asking the time constancy of the gauge fixings (3.7) we get [4] $\lambda^\mu(\tau,\vec\sigma) = \tilde\lambda^\mu(\tau) + \tilde\lambda^\mu{}_\nu(\tau) b^\nu_{\check r}(\tau)\sigma^{\check r}$, $\tilde\lambda^\mu(\tau) = -\dot x^\mu_s(\tau)$, $\tilde\lambda^{\mu\nu}(\tau) = -\tilde\lambda^{\nu\mu}(\tau) = \frac{1}{2}\sum_{\check r}[\dot b^\mu_{\check r} b^\nu_{\check r} - b^\mu_{\check r}\dot b^\nu_{\check r}](\tau)$, the Dirac Hamiltonian becomes

$$
H^F_D = \tilde\lambda^\mu(\tau)\mathcal{H}_\mu(\tau) - \frac{1}{2}\tilde\lambda^{\mu\nu}(\tau)\mathcal{H}_{\mu\nu}(\tau) + \int d^3\sigma\Big[-A_\tau(\tau,\vec\sigma)\Gamma(\tau,\vec\sigma) + \mu_\tau(\tau,\vec\sigma)\pi^\tau(\tau,\vec\sigma)\Big], \tag{3.15}
$$

and we are left with only 12 first class constraints

$$
\pi^\tau(\tau,\vec\sigma) \approx 0,
$$

$$
\begin{aligned}
\Gamma(\tau,\vec{\sigma}) &= \partial_{\check{r}}\pi^{\check{r}}(\tau,\vec{\sigma}) - \sum_{i=1}^{N}\delta^3(\vec{\sigma}-\vec{\eta}_i(\tau))Q_i(\tau) \approx 0,\\
\mathcal{H}_\mu(\tau) &= \int d^3\sigma \mathcal{H}_\mu(\tau,\vec{\sigma}) = \; p_{s\mu} - b_{\mu\tau}\Big\{\frac{1}{2}\int d^3\sigma\Big(\vec{\pi}^2(\tau,\vec{\sigma}) + \vec{B}^2(\tau,\vec{\sigma})\Big) +\\
&+ \sum_{i=1}^{N}\sqrt{m_i^2 + \big[\vec{\kappa}_i(\tau) - Q_i(\tau)\vec{A}(\tau,\vec{\eta}_i(\tau))\big]^2} +\\
&+ \frac{1}{2}\sum_{i,j=1}^{N}\delta^3(\vec{\eta}_i(\tau)-\vec{\eta}_j(\tau))\cdot\\
&\cdot\; \frac{Q_i(\tau)Q_j(\tau)\xi_i^\gamma(\tau)\xi_i^\delta(\tau)}{\sqrt{m_i^2+\vec{\kappa}_i^2(\tau)}\sqrt{m_j^2+\vec{\kappa}_j^2(\tau)}} b_{\tau\gamma}\eta_{\delta\beta}\xi_j^\alpha(\tau)\xi_j^\beta(\tau)b_{\tau\alpha} +\\
&+ i\sum_{i=1}^{N}\frac{Q_i(\tau)\xi_i^\alpha(\tau)\xi_i^\beta(\tau)}{\sqrt{m_i^2+\vec{\kappa}_i^2(\tau)}} b_{\tau\alpha}b_{\check{s}\beta}(\tau)\pi^{\check{s}}(\tau,\vec{\eta}_i(\tau)) +\\
&- \frac{i}{2}\sum_{i=1}^{N}\frac{Q_i(\tau)\xi_i^\alpha(\tau)\xi_i^\beta(\tau)}{\sqrt{m_i^2+\vec{\kappa}_i^2(\tau)}} b_{\check{u}\alpha}(\tau)b_{\check{v}\beta}(\tau)F_{\check{u}\check{v}}(\tau,\vec{\eta}_i(\tau))\Big\} +\\
&+ b_{\check{r}\mu}(\tau)\Big\{\int d^3\sigma[\vec{\pi}\times\vec{B}]_{\check{r}}(\tau,\vec{\sigma}) + \sum_{i=1}^{N}[\kappa_{i\check{r}}(\tau) - Q_i(\tau)A_{\check{r}}(\tau,\vec{\eta}_i(\tau))]\Big\} \approx 0,\\
\mathcal{H}^{\mu\nu}(\tau) &= b^\mu_{\check{r}}(\tau)\int d^3\sigma\sigma^{\check{r}}\mathcal{H}^\nu(\tau,\vec{\sigma}) - b^\nu_{\check{r}}(\tau)\int d^3\sigma\sigma^{\check{r}}\mathcal{H}^\mu(\tau,\vec{\sigma}) =\\
&= S_s^{\mu\nu}(\tau) - (b^\mu_{\check{r}}(\tau)b^\nu_\tau - b^\nu_{\check{r}}(\tau)b^\mu_\tau)\Big\{\frac{1}{2}\int d^3\sigma\sigma^{\check{r}}\Big(\vec{\pi}^2(\tau,\vec{\sigma}) + \vec{B}^2(\tau,\vec{\sigma})\Big) +\\
&+ \frac{1}{2}\sum_{i,j=1}^{N}\delta^3(\vec{\eta}_i(\tau)-\vec{\eta}_j(\tau))\cdot\\
&\cdot\; \frac{\eta_i^{\check{r}}Q_i(\tau)Q_j(\tau)\xi_i^\gamma(\tau)\xi_i^\delta(\tau)}{\sqrt{m_i^2+\vec{\kappa}_i^2(\tau)}\sqrt{m_j^2+\vec{\kappa}_j^2(\tau)}} b_{\tau\gamma}\eta_{\delta\beta}\xi_j^\alpha(\tau)\xi_j^\beta(\tau)b_{\tau\alpha} +\\
&+ \sum_{i=1}^{N}\eta_i^{\check{r}}\sqrt{m_i^2 + [\vec{\kappa}_i(\tau) - Q_i(\tau)\vec{A}(\tau,\vec{\eta}_i(\tau))]^2} +\\
&+ i\sum_{i=1}^{N}\eta_i^{\check{r}}(\tau)\frac{Q_i(\tau)\xi_i^\alpha(\tau)\xi_i^\beta(\tau)}{\sqrt{m_i^2+\vec{\kappa}_i^2(\tau)}} b_{\tau\alpha}b_{\check{s}\beta}(\tau)\pi^{\check{s}}(\tau,\vec{\eta}_i(\tau)) +\\
&- \frac{i}{2}\sum_{i=1}^{N}\eta_i^{\check{r}}(\tau)\frac{Q_i(\tau)\xi_i^\alpha(\tau)\xi_i^\beta(\tau)}{\sqrt{m_i^2+\vec{\kappa}_i^2(\tau)}} b_{\check{u}\alpha}(\tau)b_{\check{v}\beta}(\tau)F_{\check{u}\check{v}}(\tau,\vec{\eta}_i(\tau))\Big\} +\\
&+ (b^\mu_{\check{r}}(\tau)b^\nu_{\check{s}}(\tau) - b^\nu_{\check{r}}(\tau)b^\mu_{\check{s}}(\tau))\Big\{\int \; d^3\sigma\sigma^{\check{r}}[\vec{\pi}\times\vec{B}]_{\check{s}}(\tau,\vec{\sigma}) +\\
&+ \sum_{i=1}^{N}\eta_i^{\check{r}}(\tau)[\kappa_{i\check{s}}(\tau) - Q_i(\tau)A_{\check{s}}(\tau,\vec{\eta}_i(\tau))]\Big\} \approx 0. \qquad (3.16)
\end{aligned}
$$

The next step [4] is to select all the configurations of the isolated system which are timelike, namely with $p_s^2 > 0$. For them we can boost to the rest system with the standard

Wigner boost $L^\mu_{.\nu}(\overset{o}{p}_s, p_s)$ for timelike Poincaré orbits (see Appendix B) all the variables of the non-canonical basis $x^\mu_s(\tau)$, p^μ_s , $b^\mu_{\check{A}}(\tau)$, $S^{\mu\nu}_s(\tau)$, $A_{\check{A}}(\tau,\vec{\sigma})$, $\pi^{\check{A}}(\tau,\vec{\sigma})$, $\vec{\eta}_i(\tau)$, $\vec{\kappa}_i(\tau)$, $\theta_i(\tau)$, $\theta^*_i(\tau)$, $\xi^\mu_i(\tau)$ with Lorentz indices (except p^μ_s). This is a canonical transformation generated by $e^{\{.,\mathcal{F}(\tau)\}}$ with generating function

$$
\begin{aligned}
\mathcal{F}(\tau) &= \frac{1}{2}\omega(p_s) I_{\mu\nu}(p_s) S^{\mu\nu}_s(\tau),\\
I(p) &\equiv \parallel I(p)^\mu_{.\nu} \parallel = \begin{pmatrix} 0 & -\frac{p_j}{|\vec{p}|} \\ \frac{p^i}{|\vec{p}|} & 0 \end{pmatrix},\\
I_{\mu\nu}(p) &= -I_{\nu\mu}(p), \qquad\qquad I^3(p) = I(p),\\
\cosh\omega(p) &= \frac{\eta p_0}{\sqrt{p^2}}, \qquad\qquad \sinh\omega(p) = \eta\frac{|\vec{p}|}{\sqrt{p^2}},\\
L^\mu_{.\nu}(p,\overset{o}{p}) &= \exp\left[\omega(p) I(p)\right]^\mu_{.\nu} =\\
&= \left[\cosh\left(\omega(p)I(p)\right) + \sinh\left(\omega(p)I(p)\right)\right]^\mu_{.\nu} =\\
&= \left[1\!\!1 - I^2(p) + I^2(p)\cosh\omega(p) + I(p)\sinh\omega(p)\right]^\mu_{.\nu},\\
L^\mu_{.\nu}(\overset{o}{p},p) &= \exp\left[-\omega(p)I(p)\right]^\mu_{.\nu}.
\end{aligned}
\tag{3.17}
$$

Since we have $\xi^\mu_i p_{s\mu} \equiv 0$, we get $I_{\mu\nu}(p_s) S^{\mu\nu}_\xi = 0$, so that the addition of $S^{\mu\nu}_\xi$ to $S^{\mu\nu}_s$ in $\mathcal{F}$ is irrelevant.

The new non-canonical basis (with the same Dirac brackets) is

$$
\begin{aligned}
\tilde{x}^\mu_s &= x^\mu_s - \frac{1}{2}\epsilon_\nu(u(p_s))\eta_{AB}\frac{\partial\epsilon^B_\rho(u(p_s))}{\partial p_{s\mu}} S^{\nu\rho}_s =\\
&= x^\mu_s - \frac{1}{\eta_s\sqrt{p^2_s}(p^0_s+\eta_s\sqrt{p^2_s})}\Big[p_{s\nu}S^{\nu\mu}_s + \eta_s\sqrt{p^2_s}\Big(S^{0\mu}_s - S^{0\nu}_s\frac{p_{s\nu}p^\mu_s}{p^2_s}\Big)\Big] =\\
&= x^\mu_s - \frac{1}{\eta_s\sqrt{p^2_s}}\Big[\eta^\mu_A\Big(\bar{S}^{\bar{o}A}_s - \frac{\bar{S}^{Ar}_s p^r_s}{p^0_s+\eta_s\sqrt{p^2_s}}\Big) + \frac{p^\mu_s + 2\eta_s\sqrt{p^2_s}\eta^{\mu 0}}{\eta_s\sqrt{p^2_s}(p^0_s+\eta_s\sqrt{p^2_s})}\bar{S}^{\bar{o}r}_s p^r_s\Big]\\
p^\mu_s &= p^\mu_s, \quad \eta^{\check{r}}_i = \eta^{\check{r}}_i, \quad k^{\check{r}}_i = k^{\check{r}}_i, \quad A_{\check{A}} = A_{\check{A}}, \quad \pi^{\check{A}} = \pi^{\check{A}}\\
\xi^\mu_i &= \xi^\mu_i, \quad \theta^*_i = \theta^*_i, \quad \theta_i = \theta_i\\
b^A_{\check{B}} &= \epsilon^A_\mu(u(p_s)) b^\mu_{\check{B}}\\
\tilde{S}^{\mu\nu}_s &= S^{\mu\nu}_s + \frac{1}{2}\epsilon^A_\rho(u(p_s))\eta_{AB}\Big[\frac{\partial\epsilon^B_\sigma(u(p_s))}{\partial p_{s\mu}}p^\nu_s - \frac{\partial\epsilon^B_\sigma(u(p_s))}{\partial p_{s\nu}}p^\mu_s\Big]S^{\rho\sigma}_s =\\
&= S^{\mu\nu}_s + \frac{1}{\eta_s\sqrt{p^2_s}(p^0_s+\eta_s\sqrt{p^2_s})}\Big[p_{s\beta}(S^{\beta\mu}_s p^\nu_s - S^{\beta\nu}_s p^\mu_s) + \eta_s\sqrt{p^2_s}(S^{0\mu}_s p^\nu_s - S^{0\nu}_s p^\mu_s)\Big],
\end{aligned}
\tag{3.18}
$$

where $u^\mu(p_s) = p^\mu_s/\eta_s\sqrt{p^2_s} = L^\mu_o(\overset{o}{p}_s, p_s)$ [$\eta_s = \pm 1$; from now on we will select only the positive-energy branch $\eta_s = +1$].

For later use, let us introduce the spin tensors

$$
\begin{aligned}
\bar{S}^{AB}_s &= \epsilon^A_\mu(u(p_s))\epsilon^B_\nu(u(p_s)) S^{\mu\nu}_s,\\
\bar{S}^{AB}_\xi &= \epsilon^A_\mu(u(p_s))\epsilon^B_\nu(u(p_s)) S^{\mu\nu}_\xi.
\end{aligned}
\tag{3.19}
$$

Since $\xi_{i\mu} p_s^\mu \equiv 0$ implies $\xi_{i\tau} = \xi_{i\mu} u^\mu(p_s) = \xi_{i\mu} L^\mu_o(\overset{o}{p}_s, p_s) \equiv 0$, we can reduce to 3 for each particle the Grassmann variables describing the spin

$$\xi_i^r(\tau) := \epsilon^r_\mu(u(p_s))\xi_i^\mu(\tau) \qquad r = 1,2,3. \tag{3.20}$$

We get

$$\begin{aligned}
&\bar{S}_\xi^{\bar{o}B} = 0,\\
&\bar{S}_\xi^{rs} = -i\sum_{i=1}^N \epsilon^r_\mu(u(p_s))\epsilon^s_\nu(u(p_s))\xi_i^\mu\xi_i^\nu = -i\sum_{i=1}^N \xi_i^r\xi_i^s,\\
&\bar{S}_\xi^r = \frac{1}{2}\epsilon^{ruv}\bar{S}_\xi^{uv} = \sum_{i=1}^N \bar{S}^r_{i\xi},\\
&\bar{S}^r_{i\xi} = -\frac{i}{2}\epsilon^{ruv}\xi_i^u\xi_i^v.
\end{aligned} \tag{3.21}$$

The $\xi_i^r(\tau)$'s satisfy

$$\begin{aligned}
&\{\xi_i^r, \xi_j^s\}^*_D = -i\delta^{rs}\delta_{ij},\\
&\{\tilde{x}_s^\mu, \xi_i^r\}^*_D = -\frac{\partial \epsilon^r_\nu(u(p_s))}{\partial p_{s_\mu}}\xi_i^\nu.
\end{aligned} \tag{3.22}$$

If we define

$$\begin{aligned}
\hat{x}_s^\mu &\equiv \tilde{x}_s^\mu - \frac{1}{2}\epsilon^A_\nu(u(p_s))\eta_{AB}\frac{\partial \epsilon^B_\rho(u(p_s))}{\partial p_{s_\mu}} S_\xi^{\nu\rho} =\\
&= x_s^\mu - \frac{1}{2}\epsilon^A_\nu(u(p_s))\eta_{AB}\frac{\partial \epsilon^B_\rho(u(p_s))}{\partial p_{s_\mu}}\Big(S_s^{\nu\rho} + S_\xi^{\nu\rho}\Big),
\end{aligned} \tag{3.23}$$

we get

$$\begin{aligned}
&\{\hat{x}_s^\mu, p_s^\nu\}^*_D = -\eta^{\mu\nu},\\
&\{\hat{x}_s^\mu, \xi_i^r\}^*_D = 0,\\
&\{\hat{x}_s^\mu, \hat{x}_s^\nu\}^*_D = 0.
\end{aligned} \tag{3.24}$$

Therefore, with respect to the Dirac brackets $\{.,.\}^*_D$ we have obtained a basis in which $\hat{x}_s^\mu(\tau)$, p_s^μ, $A_{\check{A}}(\tau,\vec{\sigma})$, $\pi^{\check{A}}(\tau,\vec{\sigma})$, $\vec{\eta}_i(\tau)$, $\vec{\kappa}_i(\tau)$, $\xi_i^{\check{r}}(\tau)$, $\theta_i(\tau)$, $\theta_i^*(\tau)$, are canonical variables and only $b^\mu_{\check{A}}(\tau)$, $S_s^{\mu\nu}(\tau)$, are not canonical. The new canonical origin $\hat{x}_s^\mu$ of the hyperplane has the same noncovariance of the Newton-Wigner position operator and describes the *external* decoupled canonical center of mass of the isolated system. In terms of this variable we get

$$J^{\mu\nu} = \tilde{x}_s^\mu p_s^\nu - \tilde{x}_s^\nu p_s^\mu + \tilde{S}_s^{\mu\nu} + S_\xi^{\mu\nu} = \hat{L}_s^{\mu\nu} + \tilde{S}_s^{\mu\nu} + \tilde{S}_\xi^{\mu\nu},$$

$$\begin{aligned}
&\hat{L}_s^{\mu\nu} = \hat{x}_s^\mu p_s^\nu - \hat{x}_s^\nu p_s^\mu,\\
&\tilde{S}_\xi^{\mu\nu} = S_\xi^{\mu\nu} + \frac{1}{2}\epsilon^A_\rho(u(p_s))\eta_{AB}\Big[\frac{\partial \epsilon^B_\sigma(u(p_s))}{\partial p_{s_\mu}}p_s^\nu - \frac{\partial \epsilon^B_\sigma(u(p_s))}{\partial p_{s_\nu}}p_s^\mu\Big]S_\xi^{\rho\sigma},
\end{aligned}$$

$$
\begin{aligned}
&\{\hat{L}_s^{\mu\nu}, \hat{L}_s^{\alpha\beta}\}^*_D = C^{\mu\nu\alpha\beta}_{\gamma\delta}\hat{L}_s^{\gamma\delta},\\
&\{\tilde{S}_\xi^{\mu\nu}, \tilde{S}_\xi^{\alpha\beta}\}^*_D = C^{\mu\nu\alpha\beta}_{\gamma\delta}\tilde{S}_\xi^{\gamma\delta},\\
&\{\tilde{S}_s^{\mu\nu}, \tilde{S}_s^{\alpha\beta}\}^*_D = C^{\mu\nu\alpha\beta}_{\gamma\delta}\tilde{S}_s^{\gamma\delta},\\
&\{\tilde{S}_s^{\mu\nu}, \tilde{S}_\xi^{\alpha\beta}\}^*_D = \{\tilde{S}_s^{\mu\nu}, \hat{L}_s^{\alpha\beta}\}^*_D = \{\tilde{S}_\xi^{\mu\nu}, \hat{L}_s^{\alpha\beta}\}^*_D = 0,\\
&\{J^{\mu\nu}, J^{\alpha\beta}\}^*_D = C^{\mu\nu\alpha\beta}_{\gamma\delta}J^{\gamma\delta}.
\end{aligned}
\tag{3.25}
$$

As shown in Ref. [4], we can restrict ourselves to the Wigner hyperplanes $\Sigma_{W\tau}$, orthogonal to p_s^μ, i.e. with normals $l^\mu = u^\mu(p_s)$, with the gauge-fixings

$$
\begin{aligned}
&T^\mu_{\breve{A}}(\tau) = b^\mu_{\breve{A}}(\tau) - \epsilon^\mu_{A=\breve{A}}(u(p_s)) \approx 0\\
&\quad \Rightarrow b^A_{\breve{A}}(\tau) = \epsilon^A_\mu(u(p_s))b^\mu_{\breve{A}}(\tau) \approx \eta^A_{\breve{A}},
\end{aligned}
\tag{3.26}
$$

which imply the new Dirac brackets

$$
\begin{aligned}
\{A,B\}^{**}_D &= \{A,B\}^*_D - \frac{1}{4}\Big[\{A,\tilde{H}^{\gamma\delta}\}^*_D\Big(\eta_{\gamma\sigma}\epsilon^D_\gamma(u(p_s)) - \eta_{\delta\sigma}\epsilon^D_\gamma(u(p_s))\Big)\{T^\sigma_D,B\}^*_D +\\
&+ \{A,T^\sigma_D\}^*_D\Big(\eta_{\sigma\nu}\epsilon^B_\mu(u(p_s)) - \eta_{\sigma\mu}\epsilon^B_\nu(u(p_s))\Big)\{\tilde{H}^{\mu\nu},B\}^*_D\Big].
\end{aligned}
\tag{3.27}
$$

The time constancy of the gauge-fixings (3.26) implies $\tilde{\lambda}^{\mu\nu}(\tau) \approx 0$, $b^\mu_{\breve{A}}(\tau) \equiv L^\mu{}_A(p_s, \overset{o}{p}_s)$ and $\tilde{\mathcal{H}}^{\mu\nu}(\tau) \equiv 0$ (namely the determination of $S_s^{\mu\nu}$ in terms of the variables of the system). The remaining variables form a canonical basis

$$
\begin{aligned}
\{\hat{x}^\mu_s(\tau), p^\nu_s\}^{**}_D &= -\eta^{\mu\nu},\\
\{\eta^r_i(\tau), \kappa^s_j(\tau)\}^{**}_D &= \delta_{ij}\delta^{rs},\\
\{\xi^r_i(\tau), \xi^s_j(\tau)\}^{**}_D &= -i\delta^{rs}\delta_{ij},\\
\{A_A(\tau,\vec{\sigma}), \pi^B(\tau,\vec{\sigma}^{'})\}^{**}_D &= \eta^B_A\delta^3(\vec{\sigma}-\vec{\sigma}^{'}).
\end{aligned}
\tag{3.28}
$$

As shown in Ref. [4], the dependence of the gauge-fixing (3.26) on p^μ_s implies that the Lorentz-scalar indices $\breve{A} = \{\tau, \breve{r}\}$ become Wigner indices $A = \{\tau, r\}$: i) $A_{A=\tau}(\tau,\vec{\sigma})$ is a Lorentz-scalar field; ii) $A_{A=r}(\tau,\vec{\sigma})$, $\xi^r_i(\tau)$, $\eta^r_i(\tau)$, $\kappa_{ir}(\tau)$, are Wigner spin 1 3-vectors which transform with Wigner rotations under the action of *external* Minkowski Lorentz boosts.

On $\Sigma_{W\tau}$ the Poincaré generators are

$$
\begin{aligned}
&p^\mu_s, \qquad J^{\mu\nu}_s = \hat{x}^\mu_s p^\nu_s - \hat{x}^\nu_s p^\mu_s + \tilde{S}^{\mu\nu},\\
&\\
&\tilde{S}^{\mu\nu} \equiv \tilde{S}^{\mu\nu}_s + \tilde{S}^{\mu\nu}_\xi,\\
&\tilde{S}^{0i} = -\frac{\delta^{ir}\bar{S}^{rs}p^s_s}{p^0_s + \eta_s\sqrt{p^2_s}}, \qquad \tilde{S}^{ij} = \delta^{ir}\delta^{js}\bar{S}^{rs},
\end{aligned}
\tag{3.29}
$$

because one can express $\tilde{S}^{\mu\nu}$ in terms of $\bar{S}^{AB} = \epsilon^A_\mu(u(p_s))\epsilon^B_\nu(u(p_s))S^{\mu\nu}$.

Since $\tilde{H}^{\mu\nu}(\tau) \equiv 0$ implies

$$
\begin{aligned}
S^{\mu\nu}_s &= \Big(\epsilon^\mu_r(u(p_s))u^\nu(p_s) - \epsilon^\nu_r(u(p_s))u^\mu(p_s)\Big)\Big\{\frac{1}{2}\int d^3\sigma \sigma^r \Big(\vec{\pi}^2(\tau,\vec{\sigma}) + \vec{B}^2(\tau,\vec{\sigma})\Big) +\\
&+ \sum_{i=1}^N \eta^r_i(\tau)\sqrt{m_i^2 - iQ_i(\tau)\xi^u_i(\tau)\xi^v_i(\tau)F_{uv}(\tau,\vec{\eta}_i(\tau)) + [\vec{\kappa}_i(\tau) - Q_i\vec{A}(\tau,\vec{\eta}_i(\tau))]^2}\Big\} +\\
&- \Big(\epsilon^\mu_r(u(p_s))\epsilon^\nu_s(u(p_s)) - \epsilon^\nu_r(u(p_s))\epsilon^\mu_s(u(p_s))\Big)\Big\{\int d^3\sigma\sigma^r(\vec{\pi}\times\vec{B})_s(\tau,\vec{\sigma}) +\\
&+ \sum_{i=1}^N \eta^r_i(\tau)(\kappa_{is}(\tau) - Q_i(\tau)A_s(\tau,\vec{\eta}_i))\Big\}, \qquad (3.30)
\end{aligned}
$$

we get

$$
\begin{aligned}
\bar{S}^{AB}_s &= (\delta^A_r\delta^B_{\bar{o}} - \delta^B_r\delta^A_{\bar{o}})\Big[\frac{1}{2}\int d^3\sigma\sigma^r\Big(\vec{\pi}^2(\tau,\vec{\sigma}) + \vec{B}^2(\tau,\vec{\sigma})\Big) +\\
&+ \sum_{i=1}^N \eta^r_i(\tau)\sqrt{m_i^2 - iQ_i(\tau)\xi^u_i(\tau)\xi^v_i(\tau)F_{uv}(\tau,\vec{\eta}_i(\tau)) + [\vec{k}_i(\tau) - Q_i\vec{A}(\tau,\vec{\eta}_i(\tau))]^2}\Big] +\\
&- \Big(\delta^A_r\delta^B_s - \delta^B_r\delta^A_s\Big)\Big[\int d^3\sigma\sigma^r(\vec{\pi}\times\vec{B})_s(\tau,\vec{\sigma}) +\\
&+ \sum_{i=1}^N \eta^r_i(\tau)(k_{is}(\tau) - Q_i(\tau)A_s(\tau,\vec{\eta}_i(\tau)))\Big]. \qquad (3.31)
\end{aligned}
$$

However, the boosts $\bar{S}^{or}_s \equiv \bar{S}^{or}$ $[\bar{S}^{or}_\xi = 0]$ do not contribute to the previous realization of the Poincaré generators: this is the *external* Poincaré algebra in the rest-frame Wigner-covariant instant form of dynamics.

The original variables $z^\mu(\tau,\vec{\sigma})$, $\rho_\mu(\tau,\vec{\sigma})$, are reduced only to $\hat{x}^\mu_s$, p^μ_s on the Wigner hyperplane $\Sigma_{W\tau}$. On it only 6 first class constraints survive

$$
\begin{aligned}
&\pi^\tau(\tau,\vec{\sigma}) \approx 0,\\
&\Gamma(\tau,\vec{\sigma}) = \partial_r\pi^r(\tau,\vec{\sigma}) - \sum_{i=1}^N Q_i\delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) \approx 0,\\
&\mathcal{H}^\mu(\tau) = p^\mu_s - [u^\mu(p_s)\mathcal{H}_{rel}(\tau) + \epsilon^\mu_r(u(p_s))\mathcal{H}_{pr}(\tau)] =\\
&\quad = u^\mu(p_s)\mathcal{H}(\tau) + \epsilon^\mu_r(u(p_s))\mathcal{H}_{pr}(\tau) \approx 0,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{H}(\tau) &= \sqrt{p^2_s} - \mathcal{H}_{rel}(\tau) = \eta_s\sqrt{p^2_s} - \Big[\frac{1}{2}\int\ d^3\sigma\Big(\vec{\pi}^2 + \vec{B}^2\Big)(\tau,\vec{\sigma}) +\\
&+ \sum_{i=1}^N \cdot\sqrt{m_i^2 - iQ_i(\tau)\xi^r_i(\tau)\xi^s_i(\tau)F_{rs}(\tau,\vec{\eta}_i(\tau)) + [\vec{\kappa}_i(\tau) - Q_i(\tau)\vec{A}(\tau,\vec{\eta}_i(\tau))]^2}\Big] \approx 0,\\
\mathcal{H}_{pr}(\tau) &= \int\ d^3\sigma\Big[\vec{\pi}\times\vec{B}\Big]_r(\tau,\vec{\sigma}) + \sum_{i=1}^N\Big[\kappa_{ir}(\tau) - Q_i(\tau)A_r(\tau,\vec{\eta}_i(\tau))\Big] \approx 0,
\end{aligned}
$$

$$
\{\mathcal{H}^\mu(\tau),\mathcal{H}^\nu(\tau)\}^{**}_D = \int\ d^3\sigma\Big\{\Big[u^\mu(p_s)\epsilon^\nu_r(u(p_s)) - u^\nu(p_s)\epsilon^\mu_r(u(p_s))\Big]\pi^r(\tau,\vec{\sigma}) +
$$

$$- \epsilon^\mu_r(u(p_s)) F_{rs}(\tau,\vec{\sigma}) \epsilon^\nu_s(u(p_s))\Big\}\Gamma(\tau,\vec{\sigma}) \approx 0. \tag{3.32}$$

Let us remark that in $\mathcal{H}(\tau) \approx 0$ the *spin-spin* and *spin-electric field* interactions have disappeared on $\Sigma_{W\tau}$, i.e. in the inertial systems associated with the Wigner hyperplane. There is only the *spin-magnetic field* interaction

$$-iQ_i\xi^r_i\xi^s_i F_{rs}(\tau,\vec{\eta}_i(\tau)) = -2Q_i\vec{\bar{S}}_{i\xi}\cdot\vec{B}(\tau,\vec{\eta}_i(\tau)), \qquad \vec{\bar{S}}_{i\xi} \equiv -\frac{i}{2}\vec{\xi}_i\times\vec{\xi}_i, \tag{3.33}$$

like in the non-relativistic Pauli equation. This term has been put inside the square roots of the kinetic terms in analogy with the results of Section II.

Therefore, we get a kind of *relativistic Pauli Hamiltonian* describing the interaction of a positive-energy massive spinning particle belonging to the $(\frac{1}{2},0)$ representation of the Lorentz group with the electromagnetic field, whose non-relativistic limit is the pseudo-classical form of the ordinary Pauli Hamiltonian.

The constraints $\vec{\mathcal{H}}_p(\tau) \approx 0$ identify the Wigner hyperplane $\Sigma_{W\tau}$ (whose embedding in Minkowski spacetime is $z^\mu(\tau,\vec{\sigma}) = x^\mu_s(\tau) + \epsilon^\mu_r(u(p_s))\sigma^r$) with the *intrinsic rest frame* (vanishing of the total Wigner spin 1 3-momentum of the isolated system) and mean that the conjugate 3-coordinate $\vec{\sigma} = \vec{q}_+$ of the *internal* 3-center of mass of the isolated system on $\Sigma_{W\tau}$ is a *gauge* variable. The natural gauge-fixing for $\vec{\mathcal{H}}_p(\tau) \approx 0$ is $\vec{q}_+ \approx 0$; in this way the *internal* center of mass coincides with the origin of $\Sigma_{W\tau}$: $x^\mu_s(\tau) = z^\mu(\tau,\vec{\sigma}=0)$.

On $\Sigma_{W\tau}$ the Dirac Hamiltonian becomes

$$H_D = \lambda(\tau)\mathcal{H}(\tau) - \vec{\lambda}(\tau)\cdot\vec{\mathcal{H}}_p(\tau) + \int d^3\sigma\Big[-A_\tau(\tau,\vec{\sigma})\Gamma(\tau,\vec{\sigma}) + \mu_\tau(\tau,\vec{\sigma})\pi^\tau(\tau,\vec{\sigma})\Big], \tag{3.34}$$

so that the *external* canonical 4-center of mass $\hat{x}^\mu_s$ has a velocity parallel to ${p_s}^\mu$, namely it has no classical zitterbewegung just as with the Foldy-Wouthuysen mean position.

See Ref. [21] for the relativistic kinematics of the N-body problem in the rest-frame instant form of dynamics and Ref. [9] for the non-relativistic limit of $\mathcal{H}_{rel}$.

B. Dirac's observables and equations of motion.

As shown in Ref. [22], the Dirac observables of the electromagnetic field are the transverse quantities $\vec{A}_{\perp r}(\tau,\vec{\sigma})$, $\vec{\pi}^r_\perp(\tau,\vec{\sigma})$, defined by the decomposition $[\triangle = -\vec{\partial}^2]$

$$\begin{aligned}
A_r(\tau,\vec{\sigma}) &= \partial_r\eta_{em}(\tau,\vec{\sigma}) + A_{\perp r}(\tau,\vec{\sigma}), \quad \vec{\partial}\cdot\vec{A}_\perp(\tau,\vec{\sigma}) \equiv 0,\\
\pi^r(\tau,\vec{\sigma}) &= \pi^r_\perp(\tau,\vec{\sigma}) + \frac{\partial^r}{\triangle_\sigma}\Big[\Gamma(\tau,\vec{\sigma}) - \sum_{i=1}^N Q_i(\tau)\delta^3(\vec{\sigma}-\vec{\eta}_i(\tau))\Big],\\
\eta_{em}(\tau,\vec{\sigma}) &= -\frac{\vec{\partial}}{\triangle_\sigma}\cdot\vec{A}(\tau,\vec{\sigma}),
\end{aligned} \tag{3.35}$$

while the gauge variables are $A_\tau(\tau,\vec{\sigma})$ and $\eta_{em}(\tau,\vec{\sigma})$, being conjugated to the first class constraints $\pi^\tau(\tau,\vec{\sigma}) \approx 0$, $\Gamma(\tau,\vec{\sigma}) \approx 0$.

Concerning the particle variables, we have that $\kappa^r_i(\tau)$, $\theta_i(\tau)$, $\theta^*_i(\tau)$, are not gauge invariant because

$$
\begin{aligned}
\{\kappa_i^r(\tau), \Gamma(\tau,\vec{\sigma})\}^{**}_D &= Q_i \frac{\partial}{\partial \eta_i^r} \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)),\\
\{\theta_i(\tau), \Gamma(\tau,\vec{\sigma})\}^{**}_D &= i e_i \theta_i(\tau) \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)),\\
\{\theta^*_i(\tau), \Gamma(\tau,\vec{\sigma})\}^{**}_D &= -i e_i \theta^*_i(\tau) \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)).
\end{aligned}
\tag{3.36}
$$

However, the position variables $\eta_i^r(\tau)$ and the spin variables $\xi_i^r(\tau)$ are gauge invariant. The Dirac observables for the particles are obtained through a *dressing with a Coulomb cloud*

$$
\begin{aligned}
&\check{\theta}_i(\tau) = e^{i e_i \eta_{em}(\tau,\vec{\eta}_i)} \theta_i(\tau),\\
&\check{\theta}^*_i(\tau) = e^{-i e_i \eta_{em}(\tau,\vec{\eta}_i)} \theta^*_i(\tau),\\
&\check{\vec{\kappa}}_i(\tau) = \vec{\kappa}_i(\tau) - Q_i(\tau) \vec{\partial} \eta_{em}(\tau,\vec{\sigma}) \quad \Rightarrow \check{\vec{\kappa}}_i(\tau) - Q_i \vec{A}_\perp(\tau,\vec{\eta}_i) = \vec{\kappa}_i(\tau) - Q_i \vec{A}(\tau,\vec{\eta}_i).
\end{aligned}
\tag{3.37}
$$

The electric charges are gauge invariant

$$
\check{Q}_i = e_i \check{\theta}^*_i(\tau) \check{\theta}_i(\tau) = Q_i = e_i \theta^*_i(\tau) \theta_i(\tau) \qquad \Big[\dot{Q}_i(\tau) = 0 \Rightarrow Q_i(\tau) \equiv Q_i\Big].
\tag{3.38}
$$

When the Gauss law is satisfied, $\Gamma(\tau,\vec{\sigma}) = 0$, Eq.(3.35) implies

$$
\int d^3\sigma \vec{\pi}^2(\tau,\vec{\sigma}) = \int d^3\sigma \vec{\pi}^2_\perp(\tau,\vec{\sigma}) + \sum_{i\neq j}^{1...N} \frac{Q_i Q_j}{4\pi \mid \vec{\eta}_i(\tau) - \vec{\eta}_j(\tau) \mid},
\tag{3.39}
$$

so that we get

$$
\begin{aligned}
\mathcal{H}(\tau) &= \sqrt{p_s^2} - \mathcal{H}_{rel} =\\
&= \sqrt{p_s^2} - \Big\{ \frac{1}{2} \int d^3\sigma \Big(\vec{\pi}^2_\perp(\tau,\vec{\sigma}) + \vec{B}^2(\tau,\vec{\sigma}) \Big) + \sum_{i\neq j}^{1...N} \frac{Q_i Q_j}{4\pi \mid \vec{\eta}_i(\tau) - \vec{\eta}_j(\tau) \mid} +\\
&+ \sum_{i=1}^N \sqrt{m_i^2 - i Q_i \xi_i^r(\tau) \xi_i^s(\tau) F_{rs}(\tau,\vec{\eta}_i(\tau)) + [\check{\vec{\kappa}}_i(\tau) - Q_i \vec{A}_\perp(\tau,\vec{\eta}_i(\tau))]^2} \Big\} \approx 0.
\end{aligned}
\tag{3.40}
$$

We see [4] the *emergence of the Coulomb potential* from field theory and the *regularization of the Coulomb self-energy* (the $\sum_{i\neq j}$ rule) due to the Grassmann character of the electric charges, $Q_i^2 = 0$. In this way all the effects of order Q_i^2 are eliminated, but not those of order $Q_i Q_j$, $i \neq j$ [20].

Let us remark that even if we have not fixed the electromagnetic gauge but simply decoupled the gauge variables $A_\tau(\tau,\vec{\sigma})$ and $\eta_{em}(\tau,\vec{\sigma})$, this procedure is equivalent to a Wigner-covariant radiation gauge[19].

There is no odd first class constraint, because massive 2-spinors do not satisfy any spinor equation. The quantization of this Hamiltonian in the free case gives a non-local Schrödinger equation $i \frac{\partial \psi_\pm}{\partial \tau} = \pm \sqrt{m^2 + \triangle} \psi_\pm$, with the kinetic square root operator [23], for a 2-spinor,

[19]The associated natural gauge fixings would be $A_\tau(\tau,\vec{\sigma}) \approx 0$ and $\eta_{em}(\tau,\vec{\sigma}) \approx 0$.

which corresponds to the upper (or lower) part of positive (or negative) energy Dirac spinors boosted at rest [20]. These 2-spinors are parity eigenstates in the rest frame.

The three constraints defining the rest frame become

$$\vec{\mathcal{H}}_p(\tau) = \sum_{i=1}^N \check{\vec{\kappa}}_i(\tau) + \int d^3\sigma \vec{\pi}_\perp(\tau,\vec{\sigma}) \times \vec{B}(\tau,\vec{\sigma}) \approx 0, \tag{3.41}$$

and are independent of the interactions, as expected in an instant form of dynamics, like the 3-spin

$$\bar{S}^{rs}_s = \sum_{i=1}^N [\eta^r_i(\tau)\check{\kappa}^s_i(\tau) - \eta^s_i(\tau)\check{\kappa}^r_i(\tau)] + \int d^3\sigma [\sigma^r(\vec{\pi}_\perp \times \vec{B})^s - \sigma^s(\vec{\pi}_\perp \times \vec{B})^r](\tau,\vec{\sigma}),$$
$$\bar{S}^{rs} = \bar{S}^{rs}_s + \bar{S}^{rs}_\xi = \bar{S}^{rs}_s - i\sum_{i=1}^N \xi^r_i \xi^s_i. \tag{3.42}$$

The Pauli-Lubanski 4-vector and the spin Poincaré Casimir are $[\epsilon_s = \sqrt{p_s^2}]$

$$\begin{aligned} W^\mu_s &= \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} p_{s\nu} J_{s\rho\sigma} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} p_{s\nu}\tilde{S}_{\rho\sigma} = \\ &= (\vec{p}_s \cdot \vec{\bar{S}}; \epsilon_s \vec{\bar{S}} + \frac{\vec{p}_s \cdot \vec{\bar{S}}}{p^o_s + \epsilon_s}\vec{p}_s) = \\ &= \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} p_{s\nu}(\tilde{S}_{s\rho\sigma} + \tilde{S}_{\xi\rho\sigma}) := W^{(L)\mu}_s + \Sigma^\mu_s, \\ W^2_s &= -\frac{1}{2}p^2_s \tilde{S}_{\mu\nu}\tilde{S}^{\mu\nu} = -p^2_s \vec{\bar{S}}^2. \end{aligned} \tag{3.43}$$

This shows that $\vec{\bar{S}} = \vec{\bar{S}}_s + \vec{\bar{S}}_\xi$ is the *rest-frame Thomas spin*: $W_{s\mu}(p_s) = W_{s\nu} L^\nu{}_\mu(p_s; \overset{\circ}{p}_s) = (0; \epsilon_s \vec{\bar{S}})$.

As shown in Ref. [4], it is convenient to replace the *external* center-of-mass canonical coordinates $\hat{x}^\mu_s$, p^μ_s with a new basis defined by the following canonical transformation (T_s is the Lorentz-scalar time of the rest frame)

$$\begin{aligned} T_s &= \frac{p^\mu_s x_{\mu s}}{\sqrt{p^2_s}}, \qquad & \epsilon_s &= \sqrt{p^2_s}, \\ \vec{z}_s &= \sqrt{p^2_s}\Big(\hat{\vec{x}}_s - \frac{\vec{p}_s}{p^0_s}\hat{x}^0_s\Big), \qquad & \vec{k}_s &= \frac{\vec{p}_s}{\sqrt{p^2_s}}. \end{aligned} \tag{3.44}$$

The inverse canonical transformation is

[20] So that they also coincide with the corresponding parts of the positive (or negative) energy Foldy-Wouthuysen spinors boosted at rest.

$$\hat{x}^0_s = \sqrt{1-\vec{k}_s}\Big(T_s + \frac{\vec{k}_s\cdot\vec{z}_s}{\epsilon_s}\Big),$$
$$\hat{\vec{x}}_s = \frac{\vec{z}_s}{\epsilon_s} + \Big(T_s + \frac{\vec{k}_s\cdot\vec{z}_s}{\epsilon_s}\Big)\vec{k}_s,$$
$$p^0_s = \epsilon_s\sqrt{1+\vec{k}^2_s}. \tag{3.45}$$

By adding the gauge-fixing $T_s - \tau \approx 0$ [21], whose time constancy implies $\lambda(\tau) = -1$, we get a frozen phase space with the Dirac Hamiltonian $H_D = -\vec{\lambda}(\tau)\cdot\vec{\mathcal{H}}_p(\tau)$. Like in the Hamilton-Jacobi theory, to reintroduce an evolution in $\tau \equiv T_s$ one uses the energy as Hamiltonian[22] so that the Dirac Hamiltonian for the rest-frame instant form of dynamics is

$$\hat{H}_D = \mathcal{H}_{rel}(\tau) - \vec{\lambda}(\tau)\cdot\vec{\mathcal{H}}_p(\tau). \tag{3.46}$$

The effective Hamiltonian is the invariant mass M of the isolated system

$$\begin{aligned}
M = \mathcal{H}_{rel}(\tau) &= \sum_{i=1}^N \sqrt{m_i^2 - iQ_i\xi^r_i(\tau)\xi^s_i(\tau)F_{rs}(\tau,\vec{\eta}_i(\tau)) + [\check{\vec{\kappa}}_i(\tau) - Q_i\vec{A}_\perp(\tau,\vec{\eta}_i(\tau))]^2} + \\
&+ \sum_{i\neq j}^{1...N} Q_iQ_j \frac{1}{4\pi \mid \vec{\eta}_i(\tau) - \vec{\eta}_j(\tau)\mid} + \int d^3\sigma \Big(\frac{\vec{\pi}^2_\perp(\tau,\vec{\sigma}) + \vec{B}^2(\tau,\vec{\sigma})}{2}\Big) = \\
&= \sum_{i=1}^N \sqrt{m_i^2 - 2Q_i\vec{\bar{S}}_{i\xi}\cdot\vec{B}(\tau,\vec{\eta}_i) + [\check{\vec{\kappa}}_i(\tau) - Q_i\vec{A}_\perp(\tau,\vec{\eta}_i)]^2} + \\
&+ \sum_{i\neq j}^{1...N} Q_iQ_j \frac{1}{4\pi \mid \vec{\eta}_i(\tau) - \vec{\eta}_j(\tau)\mid} + \int d^3\sigma \frac{(\vec{\pi}^2_\perp + \vec{B}^2)}{2}(\tau,\vec{\sigma}).
\end{aligned} \tag{3.47}$$

See Ref. [9] for the associated Hamilton equations.

In the next Section we will modify this invariant mass to include the non-minimal couplings of the electric field identified in Section II.

---

[21] It identifies the rest-frame time T_s with the parameter τ of the foliation of Minkowski spacetime with the Wigner hyperplanes associated with the isolated system.

[22] See Ref. [24] for a different demonstration of this result.

IV. NON-MINIMAL COUPLING TO THE ELECTRIC FIELD OF THE POSITIVE-ENERGY SPINNING PARTICLE.

In this Section we modify the invariant mass (3.47) on the Wigner hyperplane to include the non-minimal coupling to the electric field of Section II. Then we define the *external* and *internal* realizations of the Poincaré group.

A. The Non-Minimal Coupling to the Electric Field.

As shown in the previous Section on the Wigner hyperplane of the rest-frame instant form we have the following four first class constraints for the description of positive-energy charged spinning particles coupled to the electromagnetic field in terms of Dirac observables corresponding to the Wigner-covariant radiation gauge

$$
\begin{aligned}
\mathcal{H}(\tau) &= \epsilon_s - \Big\{ \frac{1}{2} \int d^3\sigma \Big(\vec{\pi}_\perp^2(\tau,\vec{\sigma}) + \vec{B}^2(\tau,\vec{\sigma}) \Big) + \sum_{i\neq j}^{1...N} \frac{Q_iQ_j}{4\pi \mid \vec{\eta}_i(\tau) - \vec{\eta}_j(\tau) \mid} + \\
&+ \sum_{i=1}^N \sqrt{m_i^2 - iQ_i\xi_i^r(\tau)\xi_i^s(\tau)F_{rs}(\tau,\vec{\eta}_i(\tau)) + [\check{\vec{\kappa}}_i(\tau) - Q_i\vec{A}_\perp(\tau,\vec{\eta}_i(\tau))]^2} \Big\} \approx 0, \\
\vec{\mathcal{H}}_p(\tau) &= \sum_{i=1}^N \check{\vec{\kappa}}_i(\tau) + \int d^3\sigma \vec{\pi}_\perp(\tau,\vec{\sigma}) \times \vec{B}(\tau,\vec{\sigma}) \approx 0,
\end{aligned} \tag{4.1}
$$

The original minimal and non-minimal couplings to the electromagnetic field produced the Coulomb potential and the non-minimal coupling to the magnetic field. Now we have to introduce the modification suggested by the pseudo-classical Foldy-Wouthuysen canonical transformation of Section II to include the semiclassical non-minimal coupling to the electric field.

We see that the kinetic term of each particle is the same in Eq.(4.1) and (2.30). In Eq.(2.30) there was no choice of the electromagnetic gauge. Instead Eq.(4.1) is in the radiation gauge, so that instead of the vector potential we now have the transverse vector potential of the radiation field.

On each particle i acts: i) the radiation field; ii) the scalar Coulomb potential of the other charges. As said at the end of Section II, we have to replace $A_0(x)$ of Eq.(2.30) with just the scalar Coulomb potential

$$
Q_iV_i(\vec{\eta}_i(\tau)) = Q_i \sum_{j\neq i}^{1..N} \frac{Q_j}{4\pi|\vec{\eta}_i(\tau) - \vec{\eta}_j(\tau)|}, \tag{4.2}
$$

acting on particle i (the total Coulomb potential is $V = \frac{1}{2}\sum_{i=1}^N Q_iV_i$) while the term $\vec{\xi} \cdot (\frac{\partial \vec{A}(x)}{\partial x^0} + \vec{\partial} A_0(x))$ of Eq.(2.30) will be replaced by two terms: i) the transverse electric field $\vec{\xi} \cdot \vec{\pi}_\perp(\tau, \vec{\sigma} = \vec{\eta}_i(\tau))$ of the radiation field at the position of the particle and ii) the spatial variation of the Coulomb field of the other particles when particle i moves appearing in the combination[23]

$$\frac{iQ_i\check{\vec{\kappa}}_i(\tau)\cdot\vec{\xi}(\tau)\ \vec{\xi}(\tau)\cdot\vec{\partial}_{\vec{\eta}_i}V_i(\vec{\eta}_i(\tau))}{(m_i+\sqrt{m_i^2+\check{\vec{\kappa}}_i^2(\tau)})\sqrt{m_i^2+\check{\vec{\kappa}}_i^2(\tau)}}. \tag{4.3}$$

On the other hand, the rest-frame conditions $\vec{\mathcal{H}}_p(\tau)\approx 0$ remain the same, since they do not depend on the interaction in an instant form of the dynamics.

Therefore the modified constraints on the Wigner hyperplane, written in terms of the Dirac observables, are

$$\begin{aligned}
\mathcal{H}' &= \epsilon_s - M' = \\
&= \epsilon_s - \Big(\sum_{i=1}^N \sqrt{m_i^2 - iQ_i\xi_i^r(\tau)\xi_i^s(\tau)F_{rs}(\tau,\vec{\eta}_i(\tau)) + \check{\vec{\kappa}}_i(\tau) - Q_i\check{\vec{A}}_\perp(\tau,\vec{\eta}_i(\tau)))^2} - \\
&- i\sum_{i=1}^N \frac{Q_i\check{\vec{\kappa}}_i(\tau)\cdot\vec{\xi}_i(\tau)\,\vec{\xi}_i(\tau)\cdot\check{\vec{\pi}}_\perp(\tau,\vec{\eta}_i(\tau))}{(m_i+\sqrt{m_i^2+\check{\vec{\kappa}}_i^2(\tau)})\sqrt{m_i^2+\check{\vec{\kappa}}_i^2(\tau)}} + \\
&+ \sum_{i\neq j}\Big[\frac{Q_iQ_j}{4\pi\mid\vec{\eta}_i(\tau)-\vec{\eta}_j(\tau)\mid} - \\
&- i\frac{Q_iQ_j\check{\vec{\kappa}}_i(\tau)\cdot\vec{\xi}_i(\tau)\,\vec{\xi}_i(\tau)\cdot(\vec{\eta}_i(\tau)-\vec{\eta}_j(\tau))}{4\pi\mid\vec{\eta}_i(\tau)-\vec{\eta}_j(\tau)\mid^3(m_i+\sqrt{m_i^2+\check{\vec{\kappa}}_i^2(\tau)})\sqrt{m_i^2+\check{\vec{\kappa}}_i^2(\tau)}}\Big] + \\
&+ \int d^3\sigma\frac{1}{2}[\check{\vec{\pi}}_\perp^2+\check{\vec{B}}^2](\tau,\vec{\sigma})\Big) = \\
&= \epsilon_s - \Big(\sum_{i=1}^N\Big[\sqrt{m_i^2+(\check{\vec{\kappa}}_i(\tau)-Q_i\check{\vec{A}}_\perp(\tau,\vec{\eta}_i(\tau)))^2} - \\
&+ i\frac{Q_i\vec{\xi}_i(\tau)\times\vec{\xi}_i(\tau)\cdot\check{\vec{B}}(\tau,\vec{\eta}_i(\tau))}{2\sqrt{m_i^2+\check{\vec{\kappa}}_i^2(\tau)}} - i\frac{Q_i\check{\vec{\kappa}}_i(\tau)\cdot\vec{\xi}_i(\tau)\,\vec{\xi}_i(\tau)\cdot\check{\vec{\pi}}_\perp(\tau,\vec{\eta}_i(\tau))}{(m_i+\sqrt{m_i^2+\check{\vec{\kappa}}_i^2(\tau)})\sqrt{m_i^2+\check{\vec{\kappa}}_i^2(\tau)}}\Big] + \\
&+ \sum_{i\neq j}\Big[\frac{Q_iQ_j}{4\pi\mid\vec{\eta}_i(\tau)-\vec{\eta}_j(\tau)\mid} - \\
&- i\frac{Q_iQ_j\check{\vec{\kappa}}_i(\tau)\cdot\vec{\xi}_i(\tau)\,\vec{\xi}_i(\tau)\cdot(\vec{\eta}_i(\tau)-\vec{\eta}_j(\tau))}{4\pi\mid\vec{\eta}_i(\tau)-\vec{\eta}_j(\tau)\mid^3(m_i+\sqrt{m_i^2+\check{\vec{\kappa}}_i^2(\tau)})\sqrt{m_i^2+\check{\vec{\kappa}}_i^2(\tau)}}\Big] + \\
&+ \int d^3\sigma\frac{1}{2}[\check{\vec{\pi}}_\perp^2+\check{\vec{B}}^2](\tau,\vec{\sigma})\Big) \approx 0,
\end{aligned}$$

$$\vec{\mathcal{H}}_p(\tau) = \sum_{i=1}^N\check{\vec{\kappa}}_i(\tau) + \int d^3\sigma\,\vec{\pi}_\perp(\tau,\vec{\sigma})\times\vec{B}(\tau,\vec{\sigma}) \approx 0. \tag{4.4}$$

One can check that these constraints are still first class.

---

[23] As shown in Section VIII it is just the quantization of the consequences of this term that will produce the *spin-orbit* and *Darwin* terms in the final semiclassical Hamiltonian.

The Dirac Hamiltonian (3.46) is replaced by

$$\hat{H}_D^{'} = M^{'} - \vec{\lambda}(\tau) \cdot \vec{\mathcal{H}}_p(\tau). \tag{4.5}$$

See Subsection 2 of Appendix B for an attempt to find the constraints before the restriction to the radiation gauge and on arbitrary spacelike hypersurfaces. Since we do not know the Lagrangian, we must use Hamiltonian methods[24] for determining the energy momentum tensor T^{AB} and this requires the modified Dirac Hamiltonian on arbitrary hypersurfaces, which replaces Eq.(3.5), as input.

B. External and internal Poincaré Groups.

The *external* realization of the Poincaré algebra, given in Eq.(3.29), is

$$\begin{aligned} J^{ij}_s &:= \hat{x}^i_s p^j_s - \hat{x}^j_s p^i_s + \delta^{ir}\delta^{js}\bar{S}^{rs}, \\ J^{oi}_s &:= \hat{x}^0_s p^i_s - \hat{x}^i_s p^0_s - \frac{\delta^{ir}\bar{S}^{rs}p^s_s}{p^0_s + \eta_s\sqrt{p^2_s}}, \end{aligned} \tag{4.6}$$

where $\bar{S}^r \equiv \bar{S}^r_s + \bar{S}^r_\xi = \epsilon^{ruv}\Big(\sum_{i=1}^N \eta^u_i(\tau)\check{\kappa}^v_i(\tau) + \int d^3\sigma\, \sigma^u(\vec{\pi}_\perp \times \vec{B})^v(\tau,\vec{\sigma}) - \frac{i}{2}\sum_{i=1}^N \xi^u_i(\tau)\xi^v(\tau)\Big)$ as a consequence of Eq.(3.42).

The *internal* unfaithful realization of the Poincaré algebra has the generators [from now on we denote by M the $M^{'}$ of Eqs.(4.4), (4.5); the Green function $\vec{c}(\vec{\sigma})$ is defined in Eq.(B15) of Appendix B]

$$\begin{aligned} \mathcal{P}^\tau_{(int)} &= M = \sum_{i=1}^N \Big[\sqrt{m^2_i + (\check{\vec{\kappa}}_i(\tau) - Q_i\check{\vec{A}}_\perp(\tau,\vec{\eta}_i(\tau)))^2} - \\ &+ i\frac{Q_i\vec{\xi}_i(\tau)\times\vec{\xi}_i(\tau)\cdot\check{\vec{B}}(\tau,\vec{\eta}_i(\tau))}{2\sqrt{m^2_i + \check{\vec{\kappa}}^2_i(\tau)}} - i\frac{Q_i\check{\vec{\kappa}}_i(\tau)\cdot\vec{\xi}_i(\tau)\,\vec{\xi}_i(\tau)\cdot\check{\vec{\pi}}_\perp(\tau,\vec{\eta}_i(\tau))}{(m_i + \sqrt{m^2_i + \check{\vec{\kappa}}^2_i(\tau)})\sqrt{m^2_i + \check{\vec{\kappa}}^2_i(\tau)}}\Big] + \\ &+ \sum_{i\neq j}\Big[\frac{Q_iQ_j}{4\pi\mid\vec{\eta}_i(\tau) - \vec{\eta}_j(\tau)\mid} - \end{aligned}$$

[24]If we have the action $S = \int dtL$ of a non-singular system with configuration variables q^α (we consider a finite-dimensional case for the sake of simplicity) coupled to an external gravitational field, the energy momentum tensor is defined as $T^{\mu\nu}(z) = -\frac{2}{\sqrt{g(z)}}\frac{\delta S}{\delta g_{\mu\nu}(z)}|_{g=\eta}$. The canonical momenta and the Hamiltonian are $p_\alpha = \frac{\partial L}{\partial\,\dot{q}^\alpha}$ and $H = \frac{\partial L}{\partial\dot{q}^\alpha}\dot{q}^\alpha - L$, respectively. Note that the momenta depend on the gravitational field. Therefore, by using the notation $S_H = \int dtH = \int dtp_\alpha\dot{q}^\alpha - S$, we get $T^{\mu\nu}(z) = \frac{2}{\sqrt{g(z)}}\frac{\delta S_H}{\delta g_{\mu\nu}(z)}|_p|_{g=\eta} - \frac{2}{\sqrt{g(z)}}\int dt[\dot{q}^\alpha - \frac{\partial H}{\partial p_\alpha}]\frac{\partial T^{\mu\nu}(z)}{\partial\dot{q}^\alpha}|_{g=\eta} \stackrel{\circ}{=} \Big(\frac{2}{\sqrt{g(z)}}\frac{\delta S_H}{\delta g_{\mu\nu}(z)}|_p\Big)|_{g=\eta}$ by using the first half of Hamilton equations. This gives the Hamiltonian form of the energy momentum tensor. In the case of singular systems, the Hamiltonian has to be replaced with the Dirac Hamiltonian: the Hamiltonian form of energy momentum tensor is now dependent explicitly on the Dirac multipliers.

$$
\begin{aligned}
&- \; i\frac{Q_iQ_j\check{\vec{\kappa}}_i(\tau)\cdot\vec{\xi}_i(\tau)\,\vec{\xi}_i(\tau)\cdot(\vec{\eta}_i(\tau)-\vec{\eta}_j(\tau))}{4\pi\mid\vec{\eta}_i(\tau)-\vec{\eta}_j(\tau)\mid^3\,(m_i+\sqrt{m_i^2+\check{\vec{\kappa}}_i^2(\tau)})\sqrt{m_i^2+\check{\vec{\kappa}}_i^2(\tau)}}\Big]+\\
&+ \; \int d^3\sigma\frac{1}{2}[\check{\vec{\pi}}_\perp^2+\check{\vec{B}}^2](\tau,\vec{\sigma})=\\
&= \; \sum_{i=1}^N\Big[\sqrt{m_i^2+(\check{\vec{\kappa}}_i(\tau)-Q_i\check{\vec{A}}_\perp(\tau,\vec{\eta}_i(\tau)))^2}-\\
&\quad +i\frac{Q_i\vec{\xi}_i(\tau)\times\vec{\xi}_i(\tau)\cdot\check{\vec{B}}(\tau,\vec{\eta}_i(\tau))}{2\sqrt{m_i^2+\check{\vec{\kappa}}_i^2(\tau)}}-i\frac{Q_i\check{\vec{\kappa}}_i(\tau)\cdot\vec{\xi}_i(\tau)\,\vec{\xi}_i(\tau)\cdot\check{\vec{\pi}}_\perp(\tau,\vec{\eta}_i(\tau))}{(m_i+\sqrt{m_i^2+\check{\vec{\kappa}}_i^2(\tau)})\sqrt{m_i^2+\check{\vec{\kappa}}_i^2(\tau)}}\Big]+\\
&+ \; \sum_{i\neq j}\Big[\frac{Q_iQ_j}{4\pi\mid\vec{\eta}_i(\tau)-\vec{\eta}_j(\tau)\mid}-i\frac{Q_iQ_j\check{\vec{\kappa}}_i(\tau)\cdot\vec{\xi}_i(\tau)}{(m_i+\sqrt{m_i^2+\check{\vec{\kappa}}_i^2(\tau)})\sqrt{m_i^2+\check{\vec{\kappa}}_i^2(\tau)}}\\
&\quad \times\vec{\xi}_i(\tau)\cdot\frac{\partial}{\partial\vec{\eta}_i}\int d^3\sigma_1\,\vec{c}(\vec{\sigma}_1-\vec{\eta}_i(\tau))\cdot\vec{c}(\vec{\sigma}_1-\vec{\eta}_j(\tau))\Big]+\\
&+ \; \int d^3\sigma\frac{1}{2}[\check{\vec{\pi}}_\perp^2+\check{\vec{B}}^2](\tau,\vec{\sigma}),
\end{aligned}
$$

$$
\begin{aligned}
\vec{\mathcal{P}}_{(int)} &= \vec{\mathcal{H}}_p=\sum_{i=1}^N\check{\vec{\kappa}}_i(\tau)+\int d^3\sigma[\check{\vec{\pi}}_\perp\times\check{\vec{B}}](\tau,\vec{\sigma})\approx 0,\\
\mathcal{J}^r_{(int)} &= \bar{S}^r=\varepsilon^{rst}\bar{S}^{st}=\sum_{i=1}^N(\vec{\eta}_i(\tau)\times\check{\vec{\kappa}}_i(\tau))^r-\frac{i}{2}\sum_{i=1}^N(\vec{\xi}_i\times\vec{\xi}_i)^r+\\
&+ \int d^3\sigma\,(\vec{\sigma}\times\,[\check{\vec{\pi}}_\perp\times\check{\vec{B}}]^r(\tau,\vec{\sigma}),\\
\mathcal{K}^r_{(int)} &= \bar{S}^{\bar{o}r}_s=-\bar{S}^{r\bar{o}}_s=-\sum_{i=1}^N\vec{\eta}_i(\tau)\sqrt{m_i^2+[\check{\vec{\kappa}}_i(\tau)-Q_i\check{\vec{A}}_\perp(\tau,\;\vec{\eta}_i(\tau))]^2}+\\
&+ \sum_{i=1}^N\Big[\sum_{j\neq i}^{1..N}Q_iQ_j[\frac{1}{\triangle_{\vec{\eta}_j}}\frac{\partial}{\partial\eta_j^r}c(\vec{\eta}_i(\tau)-\vec{\eta}_j(\tau))-\eta_j^r(\tau)c(\vec{\eta}_i(\tau)-\vec{\eta}_j(\tau))]+\\
&+Q_i\int d^3\sigma\check{\pi}^r_\perp(\tau,\vec{\sigma})c(\vec{\sigma}-\;\vec{\eta}_i(\tau))\Big]-\frac{1}{2}\int d^3\sigma\sigma^r\,(\check{\vec{\pi}}_\perp^2+\check{\vec{B}}^2)(\tau,\vec{\sigma})-\\
&+ \sum_{i=1}^N\Big[i\frac{Q_i\eta_i^r(\tau)\vec{\xi}_i(\tau)\times\vec{\xi}_i(\tau)\cdot\check{\vec{B}}(\tau,\vec{\eta}_i(\tau))}{2\sqrt{m_i^2+\check{\vec{\kappa}}_i^2(\tau)}}-i\frac{Q_i\eta_i^r(\tau)\check{\vec{\kappa}}_i(\tau)\cdot\vec{\xi}_i(\tau)\,\vec{\xi}_i(\tau)\cdot\check{\vec{\pi}}_\perp(\tau,\vec{\eta}_i(\tau))}{(m_i+\sqrt{m_i^2+\check{\vec{\kappa}}_i^2(\tau)})\sqrt{m_i^2+\check{\vec{\kappa}}_i^2(\tau)}}\Big]-\\
&- i\sum_{i\neq j}\frac{Q_iQ_j\eta_i^r(\tau)\check{\vec{\kappa}}_i(\tau)\cdot\vec{\xi}_i(\tau)\,\vec{\xi}_i(\tau)\cdot(\vec{\eta}_i(\tau)-\vec{\eta}_j(\tau))}{4\pi\mid\vec{\eta}_i(\tau)-\vec{\eta}_j(\tau)\mid^3\,(m_i+\sqrt{m_i^2+\check{\vec{\kappa}}_i^2(\tau)})\sqrt{m_i^2+\check{\vec{\kappa}}_i^2(\tau)}}=\\
&= \bar{S}^{\bar{o}r}_s=-\bar{S}^{r\bar{o}}_s=-\sum_{i=1}^N\vec{\eta}_i(\tau)\sqrt{m_i^2+[\check{\vec{\kappa}}_i(\tau)-Q_i\check{\vec{A}}_\perp(\tau,\;\vec{\eta}_i(\tau))]^2}+\\
&+ \sum_{i=1}^N\Big[\sum_{j\neq i}^{1..N}Q_iQ_j[\frac{1}{\triangle_{\vec{\eta}_j}}\frac{\partial}{\partial\eta_j^r}c(\vec{\eta}_i(\tau)-\vec{\eta}_j(\tau))-\eta_j^r(\tau)c(\vec{\eta}_i(\tau)-\vec{\eta}_j(\tau))]+
\end{aligned}
$$

$$
\begin{aligned}
&+ Q_i \int d^3\sigma \check{\pi}^r_\perp(\tau,\vec{\sigma}) c(\vec{\sigma} - \vec{\eta}_i(\tau))\Big] - \frac{1}{2}\int d^3\sigma \sigma^r\, (\check{\vec{\pi}}^2_\perp + \check{\vec{B}}^2)(\tau,\vec{\sigma}) - \\
&+ \sum_{i=1}^N \Big[i \frac{Q_i \eta_i^r(\tau) \vec{\xi}_i(\tau) \times \vec{\xi}_i(\tau) \cdot \check{\vec{B}}(\tau,\vec{\eta}_i(\tau))}{2\sqrt{m_i^2 + \check{\vec{\kappa}}_i^2(\tau)}} - i \frac{Q_i \eta_i^r(\tau) \check{\vec{\kappa}}_i(\tau) \cdot \vec{\xi}_i(\tau)\, \vec{\xi}_i(\tau) \cdot \check{\vec{\pi}}_\perp(\tau,\vec{\eta}_i(\tau))}{(m_i + \sqrt{m_i^2 + \check{\vec{\kappa}}_i^2(\tau)})\sqrt{m_i^2 + \check{\vec{\kappa}}_i^2(\tau)}} \Big] - \\
&- \; i \sum_{i\neq j}^{1..N} \frac{Q_i Q_j \eta_i^r(\tau) \check{\vec{\kappa}}_i(\tau) \cdot \vec{\xi}_i(\tau)}{(m_i + \sqrt{m_i^2 + \check{\vec{\kappa}}_i^2(\tau)})\sqrt{m_i^2 + \check{\vec{\kappa}}_i^2(\tau)}} \\
&\quad \vec{\xi}_i(\tau) \cdot \frac{\partial}{\partial \vec{\eta}_i} \int d^3\sigma_1\, \vec{c}(\vec{\sigma}_1 - \vec{\eta}_i(\tau)) \cdot \vec{c}(\vec{\sigma}_1 - \vec{\eta}_j(\tau)).
\end{aligned}
\tag{4.7}
$$

As said in Refs. [5,10], the natural gauge fixings for the rest-frame conditions $\vec{\mathcal{P}}_{(int)} \approx 0$ are $\vec{\mathcal{K}}_{(int)} \approx 0$. These conditions identify the *internal* 3-center of energy of Møller, which weakly coincides with the *internal* canonical 3-center of mass $\vec{q}_+$ due to $\vec{\mathcal{P}}_{(int)} \approx 0$. Therefore the gauge fixings $\vec{\mathcal{K}}_{(int)} \approx 0$ are equivalent to $\vec{q}_+ \approx 0$: they put the *internal* 3-center of mass in the origin $x^\mu_s(\tau) = z^\mu(\tau,\vec{0})$ of the 3-coordinates in each Wigner hyperplane. The gauge fixings $\vec{q}_+ \approx 0$ imply $\vec{\lambda}(\tau) = 0$.

V. THE EQUATIONS OF MOTION FOR N CHARGED POSITIVE-ENERGY SPINNING PARTICLES PLUS THE ELECTROMAGNETIC FIELD.

The modified Dirac Hamiltonian (4.5), (4.7) in the rest-frame instant form on Wigner hyperplanes and with $T_s \equiv \tau$ can be written in the form

$$
\begin{aligned}
\hat{H}_D &= M - \vec{\lambda}(\tau)\cdot\vec{\mathcal{H}}_p(\tau) = \\
&= \sum_{i=1}^N \Big[\sqrt{m_i^2 + (\check{\vec{\kappa}}_i(\tau) - Q_i\check{\vec{A}}_\perp(\tau,\vec{\eta}_i(\tau)))^2} + \frac{\mathcal{R}_i(\tau)}{2\sqrt{m_i^2+\check{\vec{\kappa}}_i^2(\tau)}}\Big] + \\
&+ \sum_{i\neq j}\frac{Q_iQ_j}{4\pi\mid\vec{\eta}_i(\tau)-\vec{\eta}_j(\tau)\mid}\Big[1 - \\
&- i\frac{\check{\vec{\kappa}}_i(\tau)\cdot\vec{\xi}_i(\tau)\,\vec{\xi}_i(\tau)\cdot(\vec{\eta}_i(\tau)-\vec{\eta}_j(\tau))}{\mid\vec{\eta}_i(\tau)-\vec{\eta}_j(\tau)\mid^2\,(m_i+\sqrt{m_i^2+\check{\vec{\kappa}}_i^2(\tau)})\sqrt{m_i^2+\check{\vec{\kappa}}_i^2(\tau)}}\Big] + \\
&+ \int d^3\sigma\frac{1}{2}[\check{\vec{\pi}}_\perp^2 + \check{\vec{B}}^2](\tau,\vec{\sigma}) - \vec{\lambda}(\tau)\cdot\vec{\mathcal{H}}_p(\tau) = \\
&= \sum_{i=1}^N \sqrt{m_i^2 + \mathcal{R}_i(\tau) + (\check{\vec{\kappa}}_i(\tau) - Q_i\check{\vec{A}}_\perp(\tau,\vec{\eta}_i(\tau)))^2} + \\
&+ \sum_{i\neq j}\frac{Q_iQ_j}{4\pi\mid\vec{\eta}_i(\tau)-\vec{\eta}_j(\tau)\mid}\Big[1 - \\
&- i\frac{\check{\vec{\kappa}}_i(\tau)\cdot\vec{\xi}_i(\tau)\,\vec{\xi}_i(\tau)\cdot(\vec{\eta}_i(\tau)-\vec{\eta}_j(\tau))}{\mid\vec{\eta}_i(\tau)-\vec{\eta}_j(\tau)\mid^2\,(m_i+\sqrt{m_i^2+\check{\vec{\kappa}}_i^2(\tau)})\sqrt{m_i^2+\check{\vec{\kappa}}_i^2(\tau)}}\Big] + \\
&+ \int d^3\sigma\frac{1}{2}[\check{\vec{\pi}}_\perp^2 + \check{\vec{B}}^2](\tau,\vec{\sigma}) - \vec{\lambda}(\tau)\cdot\vec{\mathcal{H}}_p(\tau),
\end{aligned}
\tag{5.1}
$$

where we introduced the quantity

$$
\begin{aligned}
\mathcal{R}_i(\tau) &= +iQ_i\vec{\xi}_i(\tau)\times\vec{\xi}_i(\tau)\cdot\check{\vec{B}}(\tau,\vec{\eta}_i(\tau)) - 2i\frac{Q_i\check{\vec{\kappa}}_i(\tau)\cdot\vec{\xi}_i(\tau)\,\vec{\xi}_i(\tau)\cdot\check{\vec{\pi}}_\perp(\tau,\vec{\eta}_i(\tau))}{m_i+\sqrt{m_i^2+\check{\vec{\kappa}}_i^2(\tau)}} = \\
&= -2Q_i\Big[\bar{\vec{S}}_{i\xi}(\tau)\cdot\check{\vec{B}}(\tau,\vec{\eta}_i(\tau)) - \frac{\check{\vec{\kappa}}_i(\tau)\cdot\check{\vec{\pi}}_\perp(\tau,\vec{\eta}_i(\tau))\times\bar{\vec{S}}_{i\xi}(\tau)}{m_i+\sqrt{m_i^2+\check{\vec{\kappa}}_i^2(\tau)}}\Big] = \\
&= -2Q_i\bar{\vec{S}}_{i\xi}(\tau)\cdot\Big[\check{\vec{B}}(\tau,\vec{\eta}_i(\tau)) + \frac{\check{\vec{\pi}}_\perp(\tau,\vec{\eta}_i(\tau))\times\check{\vec{\kappa}}_i(\tau)}{m_i+\sqrt{m_i^2+\check{\vec{\kappa}}_i^2(\tau)}}\Big].
\end{aligned}
\tag{5.2}
$$

Analogously the modified Coulomb potential may be written as

$$
\begin{aligned}
&\sum_{i\neq j}\frac{Q_iQ_j}{4\pi\mid\vec{\eta}_i(\tau)-\vec{\eta}_j(\tau)\mid}\Big[1 + \\
&+\frac{\check{\vec{\kappa}}_i(\tau)\cdot(\vec{\eta}_i(\tau)-\vec{\eta}_j(\tau))\times\bar{\vec{S}}_{i\xi}(\tau)}{\mid\vec{\eta}_i(\tau)-\vec{\eta}_j(\tau)\mid^2\,(m_i+\sqrt{m_i^2+\check{\vec{\kappa}}_i^2(\tau)})\sqrt{m_i^2+\check{\vec{\kappa}}_i^2(\tau)}}\Big].
\end{aligned}
\tag{5.3}
$$

By using the Hamilton-Dirac equations of motion, $\dot{O} \circeq \{O, \hat{H}_D\}$, we obtain (where '$\circeq$' means evaluated on the solutions of the equations of motion)

$$
\begin{aligned}
\dot{\vec{\eta}}_l(\tau) &\circeq \frac{\check{\vec{\kappa}}_l(\tau) - Q_l \check{\vec{A}}_\perp(\tau, \vec{\eta}_l(\tau)) + \frac{1}{2}\frac{\partial \mathcal{R}_l(\tau)}{\partial \check{\vec{\kappa}}_l}}{\sqrt{m_l^2 + \mathcal{R}_l + (\check{\vec{\kappa}}_l(\tau) - Q_l \check{\vec{A}}_\perp(\tau, \vec{\eta}_l(\tau)))^2}} - \vec{\lambda}(\tau) - \\
&- i \sum_{j\neq l}^N \frac{Q_l Q_j \vec{\xi}_l(\tau)\, \vec{\xi}_l(\tau) \cdot (\vec{\eta}_l(\tau) - \vec{\eta}_j(\tau))}{4\pi \mid \vec{\eta}_l(\tau) - \vec{\eta}_j(\tau) \mid^3 (m_l + \sqrt{m_l^2 + \check{\vec{\kappa}}_l^2(\tau)})\sqrt{m_l^2 + \check{\vec{\kappa}}_l^2(\tau)}} + \\
&+ i \sum_{j\neq l}^N \frac{Q_l Q_j \check{\vec{\kappa}}_l(\tau) \cdot \vec{\xi}_l(\tau)\, \vec{\xi}_l(\tau) \cdot (\vec{\eta}_l(\tau) - \vec{\eta}_j(\tau)) \check{\vec{\kappa}}_l(\tau)}{4\pi \mid \vec{\eta}_l(\tau) - \vec{\eta}_j(\tau) \mid^3 (m_l + \sqrt{m_l^2 + \check{\vec{\kappa}}_l^2(\tau)})(m_l^2 + \check{\vec{\kappa}}_l^2(\tau))^{3/2}} + \\
&+ i \sum_{j\neq l}^N \frac{Q_l Q_j \check{\vec{\kappa}}_l(\tau) \cdot \vec{\xi}_l(\tau)\, \vec{\xi}_l(\tau) \cdot (\vec{\eta}_l(\tau) - \vec{\eta}_j(\tau)) \check{\vec{\kappa}}_l}{4\pi \mid \vec{\eta}_l(\tau) - \vec{\eta}_j(\tau) \mid^3 (m_l + \sqrt{m_l^2 + \check{\vec{\kappa}}_l^2(\tau)})^2 (m_l^2 + \check{\vec{\kappa}}_l^2(\tau))} = \\
&= \frac{\check{\vec{\kappa}}_l(\tau)}{\sqrt{m_l^2 + \mathcal{R}_l(\tau) + (\check{\vec{\kappa}}_l(\tau) - Q_l \check{\vec{A}}_\perp(\tau, \vec{\eta}_l(\tau)))^2}} + \\
&+ \frac{-Q_l \check{\vec{A}}_\perp(\tau, \vec{\eta}_l(\tau)) + \frac{1}{2}\frac{\partial \mathcal{R}_l(\tau)}{\partial \check{\vec{\kappa}}_l}}{\sqrt{m_l^2 + \check{\vec{\kappa}}_l^2(\tau)}} - \vec{\lambda}(\tau) - \\
&- i \sum_{j\neq l}^N \frac{Q_l Q_j \vec{\xi}_l(\tau)\, \vec{\xi}_l(\tau) \cdot (\vec{\eta}_l(\tau) - \vec{\eta}_j(\tau))}{4\pi \mid \vec{\eta}_l(\tau) - \vec{\eta}_j(\tau) \mid^3 (m_l + \sqrt{m_l^2 + \check{\vec{\kappa}}_l^2(\tau)})\sqrt{m_l^2 + \check{\vec{\kappa}}_l^2(\tau)}} + \\
&+ i \sum_{j\neq l}^N \frac{Q_l Q_j \check{\vec{\kappa}}_l(\tau) \cdot \vec{\xi}_l(\tau)\, \vec{\xi}_l(\tau) \cdot (\vec{\eta}_l(\tau) - \vec{\eta}_j(\tau)) \check{\vec{\kappa}}_l(\tau)}{4\pi \mid \vec{\eta}_l(\tau) - \vec{\eta}_j(\tau) \mid^3 (m_l + \sqrt{m_l^2 + \check{\vec{\kappa}}_l^2(\tau)})(m_l^2 + \check{\vec{\kappa}}_l^2(\tau))^{3/2}} + \\
&+ i \sum_{j\neq l}^N \frac{Q_l Q_j \check{\vec{\kappa}}_l(\tau) \cdot \vec{\xi}_l(\tau)\, \vec{\xi}_l(\tau) \cdot (\vec{\eta}_l(\tau) - \vec{\eta}_j(\tau)) \check{\vec{\kappa}}_l(\tau)}{4\pi \mid \vec{\eta}_l(\tau) - \vec{\eta}_j(\tau) \mid^3 (m_l + \sqrt{m_l^2 + \check{\vec{\kappa}}_l^2(\tau)})^2 (m_l^2 + \check{\vec{\kappa}}_l^2(\tau))},
\end{aligned}
$$

$$\Downarrow$$

$$
Q_i \dot{\vec{\eta}}_i(\tau) \circeq Q_i \frac{\check{\vec{\kappa}}_i(\tau)}{\sqrt{m_i^2 + \check{\vec{\kappa}}_i^2(\tau)}}, \qquad Q_i \sqrt{1 - \dot{\vec{\eta}}_i^2(\tau)} \circeq Q_i \frac{|\check{\vec{\kappa}}_i(\tau)|}{\sqrt{m_i^2 + \check{\vec{\kappa}}_i^2(\tau)}}, \tag{5.4}
$$

and

$$
\begin{aligned}
\dot{\check{\vec{\kappa}}}_l(\tau) &= - \sum_{k\neq l} \frac{Q_l Q_k (\vec{\eta}_l(\tau) - \vec{\eta}_k(\tau))}{4\pi \mid \vec{\eta}_l(\tau) - \vec{\eta}_k(\tau) \mid^3} - \\
&- \sum_{j\neq l} i \frac{Q_l Q_j \check{\vec{\kappa}}_l(\tau) \cdot \vec{\xi}_l(\tau)\, \vec{\xi}_l(\tau) \cdot \Big[|\vec{\eta}_l(\tau) - \vec{\eta}_j(\tau)|^2 I - 3(\vec{\eta}_l(\tau) - \vec{\eta}_j(\tau))(\vec{\eta}_l(\tau) - \vec{\eta}_j(\tau)) \Big]}{4\pi \mid \vec{\eta}_l(\tau) - \vec{\eta}_j(\tau) \mid^5 (m_l + \sqrt{m_l^2 + \check{\vec{\kappa}}_l^2(\tau)})\sqrt{m_l^2 + \check{\vec{\kappa}}_l^2(\tau)}} + \\
&+ \frac{1}{\sqrt{m_l^2 + \mathcal{R}_l(\tau) + (\check{\vec{\kappa}}_l(\tau) - Q_l \check{\vec{A}}_\perp(\tau, \vec{\eta}_l(\tau)))^2}} \\
&\quad \Big[(\check{\vec{\kappa}}_l(\tau) - Q_l \check{\vec{A}}_\perp(\tau, \vec{\eta}_l(\tau))_u Q_l \frac{\partial}{\partial \vec{\eta}_l} \check{A}_\perp^u(\tau, \vec{\eta}_i(\tau)) + \frac{1}{2}\frac{\partial \mathcal{R}_l(\tau)}{\partial \vec{\eta}_l} \Big] =
\end{aligned}
$$

$$= - \sum_{k\not= l} \frac{Q_l Q_k (\vec \eta_l(\tau) - \vec \eta_k(\tau))}{4\pi \mid \vec \eta_l(\tau) - \vec \eta_k(\tau) \mid^3} + \frac{Q_l \check{\vec \kappa}_l(\tau)_u \frac{\partial}{\partial\, \vec \eta_l} \check{A}^u_{\perp}(\tau, \vec \eta_l(\tau))}{\sqrt{m_l^2 + \check{\vec \kappa}_l^2(\tau)}} + \frac{\frac{1}{2}\frac{\partial \mathcal{R}_l(\tau)}{\partial \vec \eta_l}}{\sqrt{m_l^2 + \check{\vec \kappa}_l^2(\tau)}} -$$

$$- Q_l \sum_{j\not= l} i \frac{Q_j \check{\vec \kappa}_l \cdot \vec \xi_l \vec \xi_l \cdot \Big[|\vec \eta_l(\tau) - \vec \eta_j(\tau)|^2 I - 3(\vec \eta_l(\tau) - \vec \eta_j(\tau))(\vec \eta_l(\tau) - \vec \eta_j(\tau)) \Big]}{4\pi \mid \vec \eta_l(\tau) - \vec \eta_j(\tau) \mid^5 \, (m_l + \sqrt{m_l^2 + \check{\vec \kappa}_l^2(\tau)}) \sqrt{m_l^2 + \check{\vec \kappa}_l^2(\tau)}}. \tag{5.5}$$

To these Hamilton-Dirac equations we must add the rest-frame condition

$$\sum_{i=1}^N \check{\vec \kappa}_i(\tau) + \int d^3\sigma [\check{\vec \pi}_{\perp} \times \check{\vec B}](\tau, \vec \sigma) \approx 0, \tag{5.6}$$

whose natural gauge fixing, implying $\vec \lambda(\tau) = 0$, is $\vec{\mathcal{K}}_{(int)} \approx 0$, as mentionied in the previous Section.

The equations of motion for the Grassmann variables and the spins are

$$\begin{aligned}
\dot{\vec \xi}_l(\tau) &\, {\buildrel \circ \over =}\, \frac{-i}{\sqrt{m_l^2 + \check{\vec \kappa}_l^2(\tau)}} \frac{1}{2} \frac{\partial \mathcal{R}_l(\tau)}{\partial \vec \xi_l} - \\
&\quad - Q_l \sum_{j\not= l} \frac{Q_j [\check{\vec \kappa}_l \vec \xi_l \cdot (\vec \eta_l(\tau) - \vec \eta_j(\tau))] - \check{\vec \kappa}_l \cdot \vec \xi_l (\vec \eta_l(\tau) - \vec \eta_j(\tau))}{4\pi \mid \vec \eta_l(\tau) - \vec \eta_j(\tau) \mid^3 \, (m_l + \sqrt{m_l^2 + \check{\vec \kappa}_l^2(\tau)}) \sqrt{m_l^2 + \check{\vec \kappa}_l^2(\tau)}}, \\
\dot{\bar S}^r_{i\xi}(\tau) &\, {\buildrel \circ \over =}\, \Omega_i^{rs}(\tau)\, \bar S^s_{i\xi}(\tau), \\
\dot{\bar S}^r_{\xi}(\tau) &= \sum_{i=1}^N \dot{\bar S}^r_{i\xi}(\tau) \, {\buildrel \circ \over =}\, \sum_{i=1}^N \Omega_i^{rs}(\tau) \bar S^s_{i\xi}(\tau), \\
\Omega_i^{rs}(\tau) &= \frac{Q_i}{\sqrt{m_i^2 + \check{\vec \kappa}_i^2}} \Big[\epsilon^{rst} \check{\vec B}_{\perp}(\tau, \vec \eta_i(\tau)) + \\
&\quad + \sum_{j\not= i} \frac{Q_j}{(m_i + \sqrt{m_i + \vec \kappa_i^2})(\sqrt{m_i + \vec \kappa_i^2})} \\
&\quad \Big(\check{\kappa}_i^r(\tau) (\check{\pi}^s_{\perp}(\tau, \vec \eta_i(\tau)) + \frac{Q_j}{4\pi \eta_{ij}^3} \eta^s_{ij}(\tau)) - \check{\kappa}_i^s(\tau) (\check{\pi}^r_{\perp}(\tau, \vec \eta_i(\tau)) + \frac{Q_j}{4\pi \eta_{ij}^3} \eta^r_{ij}(\tau)) \Big) \Big].
\end{aligned} \tag{5.7}$$

By using

$$\{ \check{A}^r_{\perp}(\tau, \vec \sigma), \check{\vec \pi}_{\perp s}(\tau, \vec \eta_i(\tau)) \} = \delta_{rs} \delta^3(\vec \sigma - \vec \eta_i\, (\tau))\, , \tag{5.8}$$

the Hamilton-Dirac equations for the transverse fields are [$P^{rs}_{\perp}(\vec \sigma) = \delta^{rs} + \frac{\partial^r \partial^s}{\triangle}$ is the transverse projector]

$$\begin{aligned}
\dot{\check{A}}_{\perp r}(\tau, \vec \sigma) \, &{\buildrel \circ \over =}\, - \check{\pi}_{\perp r}(\tau, \vec \sigma) - \sum_{i=1}^N P_{\perp rs}(\vec \sigma) \frac{i Q_i \check{\vec \kappa}_i(\tau) \cdot \vec \xi_i(\tau) \xi_i^s(\tau) \delta^3(\vec \sigma - \vec \eta_i\, (\tau))}{(m_i + \sqrt{m_i^2 + \check{\vec \kappa}_i^2(\tau)}) \sqrt{m_i^2 + \check{\vec \kappa}_i^2(\tau)}} - \\
&\quad - [\vec \lambda(\tau) \cdot \vec \partial] \check{A}_{\perp r}(\tau, \vec \sigma),
\end{aligned}$$

$$
\begin{aligned}
\dot{\check{\pi}}^r_\perp(\tau,\vec\sigma) &\stackrel{\circ}{=} \Delta \check{A}^r_\perp(\tau,\vec\sigma) - [\vec\lambda(\tau)\cdot\vec\partial]\check{\pi}^r_\perp(\tau,\vec\sigma)\\
&+ \sum_i \Big[i\frac{Q_i\xi^r_i(\tau)\,\vec\xi_i(\tau)\cdot\vec\partial\delta^3(\vec\sigma-\vec\eta_i(\tau))}{\sqrt{m_i^2+\check{\vec\kappa}_i^2(\tau)}} - Q_i P^{rs}_\perp(\vec\sigma)\dot\eta^s_i(\tau)\delta^3(\vec\sigma-\vec\eta_i(\tau))\Big] =\\
&= -\frac{\partial^2\check{A}^r_\perp}{\partial\tau^2} - \sum_{i=1}^N iQ_iP_{\perp rs}(\vec\sigma)\Big[\dot{\check{\vec\kappa}}_l(\tau)\cdot\Big(\frac{\bar 1}{(m_i+\sqrt{m_i^2+\check{\vec\kappa}_i^2(\tau)})\sqrt{m_i^2+\check{\vec\kappa}_i^2(\tau)}} -\\
&- \frac{\check{\vec\kappa}_i(\tau)\,\check{\vec\kappa}_i(\tau)}{(m_i+\sqrt{m_i^2+\check{\vec\kappa}_i^2(\tau)})(\sqrt{m_i^2+\check{\vec\kappa}_i^2(\tau)})^3} -\\
&- \frac{\check{\vec\kappa}_i(\tau)\,\check{\vec\kappa}_i(\tau)}{(m_i+\sqrt{m_i^2+\check{\vec\kappa}_i^2(\tau)})^2(\sqrt{m_i^2+\check{\vec\kappa}_i^2(\tau)})^2}\Big)\cdot\vec\xi_i(\tau)\xi^s_i(\tau)\delta^3(\vec\sigma-\vec\eta_i(\tau)) +\\
&+ \frac{\check{\vec\kappa}_i(\tau)\cdot\dot{\vec\xi}_i(\tau)\xi^s_i(\tau)\delta^3(\vec\sigma-\vec\eta_i(\tau))}{(m_i+\sqrt{m_i^2+\check{\vec\kappa}_i^2(\tau)})\sqrt{m_i^2+\check{\vec\kappa}_i^2(\tau)}} +\\
&+ \frac{\check{\vec\kappa}_i(\tau)\cdot\vec\xi_i(\tau)\dot\xi^s_i(\tau)\delta^3(\vec\sigma-\vec\eta_i(\tau))}{(m_i+\sqrt{m_i^2+\check{\vec\kappa}_i^2(\tau)})\sqrt{m_i^2+\check{\vec\kappa}_i^2(\tau)}} -\\
&- \frac{\check{\vec\kappa}_i(\tau)\cdot\vec\xi_i(\tau)\ \xi^s_i(\tau)\dot{\vec\eta}_l(\tau)\cdot\vec\partial_\sigma\delta^3(\vec\sigma-\vec\eta_i(\tau))}{(m_i+\sqrt{m_i^2+\check{\vec\kappa}_i^2(\tau)})\sqrt{m_i^2+\check{\vec\kappa}_i^2(\tau)}}\Big].
\end{aligned}
\tag{5.9}
$$

When we eliminate $\dot{\check{\vec\kappa}}_l(\tau)$, $\dot{\vec\xi}_i(\tau)$ with Eqs.(5.5), (5.7), by virtue of the Grassmann truncations due to $Q_i^2=0$ we arrive at the following wave equation for the transverse potential

$$
\begin{aligned}
&\frac{\partial^2\check{A}^r_\perp(\tau,\vec\sigma)}{\partial\tau^2} + \Delta\check{A}^r_\perp(\tau,\vec\sigma) - [\vec\lambda(\tau)\cdot\vec\partial]\check{\pi}^r_\perp(\tau,\vec\sigma) =\\
&= \sum_i Q_i\Big[P^{rs}_\perp(\vec\sigma)\dot\eta^s_i(\tau)\delta^3(\vec\sigma-\vec\eta_i(\tau)) -\\
&- i\frac{\xi^r_i(\tau)\vec\xi_i(\tau)\cdot\vec\partial\delta^3(\vec\sigma-\vec\eta_i(\tau))}{\sqrt{m_i^2+\check{\vec\kappa}_i^2(\tau)}} +\\
&+ P^{rs}_\perp(\vec\sigma)i\frac{\check{\vec\kappa}_i(\tau)\cdot\vec\xi_i(\tau)\ \xi^s_i(\tau)\dot{\vec\eta}_i(\tau)\cdot\vec\partial_\sigma\delta^3(\vec\sigma-\vec\eta_i(\tau))}{(m_i+\sqrt{m_i^2+\check{\vec\kappa}_i^2(\tau)})\sqrt{m_i^2+\check{\vec\kappa}_i^2(\tau)}}\Big].
\end{aligned}
\tag{5.10}
$$

Expressed in terms of particle velocities the right hand side becomes

$$
\begin{aligned}
&= \sum_i Q_i\Big[P^{rs}_\perp(\vec\sigma)\dot\eta^s_i(\tau)\delta^3(\vec\sigma-\vec\eta_i(\tau)) -\\
&- i\frac{\sqrt{(1-\dot{\vec\eta}_i^2}\xi^r_i(\tau)\vec\xi_i(\tau)\cdot\vec\partial\delta^3(\vec\sigma-\vec\eta_i(\tau))}{m_i} +\\
&+ P^{rs}_\perp(\vec\sigma)i\frac{\sqrt{1-\dot{\vec\eta}_i^2(\tau)}\ \dot{\vec\eta}_i(\tau)\cdot\vec\xi_i(\tau)\xi^s_i(\tau)\dot{\vec\eta}_i(\tau)\cdot\vec\partial_\sigma\delta^3(\vec\sigma-\vec\eta_i(\tau))}{m_i(\sqrt{1-\dot{\vec\eta}_i^2(\tau)}+1)}\Big].
\end{aligned}
\tag{5.11}
$$

Let us now choose a gauge (the natural one is $\vec{\mathcal{K}}_{(int)}\approx 0$) implying $\vec\lambda(\tau)=0$. In this gauge we obtain

$$\Box A^r_\perp(\tau,\vec\sigma) = (\frac{\partial^2}{\partial\tau^2} - \vec\partial^2) A^r_\perp(\tau,\vec\sigma) =$$
$$= \sum_{i=1}^N Q_i \Big[\frac{i\dot{\vec\eta}_i(\tau)\cdot\vec\partial_\sigma\, \dot{\vec\eta}_i(\tau)\cdot\vec\xi_i(\tau)\,\xi^s_i(\tau)\sqrt{1-\dot{\vec\eta}^2_i(\tau)}}{(1+\sqrt{1-\dot{\vec\eta}^2_i(\tau)})m_i} -$$
$$- \frac{i\xi^s_i(\tau)\vec\xi_i(\tau)\cdot\vec\partial_\sigma\sqrt{1-\dot{\vec\eta}^2_i(\tau)}}{m_i} + \dot\eta^s_i(\tau)\Big](\delta^{sr} + \frac{\partial^s\partial^r}{\Delta})\delta^3(\vec\sigma - \vec\eta_i(\tau)) =$$
$$: = j^r_\perp(\tau,\vec\sigma) = P^{rs}_\perp(\vec\sigma)\sum_{i=1}^N Q_i \mathbf{V}^s_i(\tau)\delta^3(\vec\sigma-\vec\eta_i(\tau)). \qquad (5.12)$$

We see that the effective current $\vec j_\perp(\tau,\vec\sigma)$ has the structure of a *pole-dipole*. It is convenient to define an effective particle velocity operator (using boldface) $\vec{\mathbf{V}}_i$

$$\mathbf{V}^s_i(\tau) = \dot\eta^s_i(\tau) - \frac{i\sqrt{1-\dot{\vec\eta}^2_i(\tau)}\xi^s_i(\tau)\vec\xi_i(\tau)\cdot\vec\partial_\sigma}{m_i} + \frac{i\sqrt{1-\dot{\vec\eta}^2_i(\tau)}\dot{\vec\eta}_i(\tau)\cdot\vec\xi_i(\tau)\xi^s_i(\tau)\dot{\vec\eta}_i(\tau)\cdot\vec\partial_\sigma}{m_i(1+\sqrt{1-\dot{\vec\eta}^2_i(\tau)})},$$

$$\Downarrow$$

$$Q_i\vec{\mathbf{V}}_i(\tau) \circeq \frac{Q_i}{\sqrt{m_i^2+\check{\vec\kappa}^2_i(\tau)}}\Big[\check{\vec\kappa}_i(\tau) - \frac{|\check{\vec\kappa}_i(\tau)|}{m_i}\vec{\bar S}_{i\xi}(\tau)\times\vec\partial_\sigma +$$
$$+ \frac{|\check{\vec\kappa}_i(\tau)|\ \vec{\bar S}_{i\xi}(\tau)\times\check{\vec\kappa}_i(\tau)\ \check{\vec\kappa}_i(\tau)\cdot\vec\partial_\sigma}{m_i\sqrt{m_i^2+\check{\vec\kappa}^2_i(\tau)}(|\check{\vec\kappa}_i(\tau)| + \sqrt{m_i^2+\check{\vec\kappa}^2_i(\tau)})}\Big]. \qquad (5.13)$$

In the last line the operators $\vec{\mathbf{V}}_i$ have been reexpressed in terms of the momenta by using Eqs.(5.4).

We point out that defining $\vec\beta_i(\tau) = \dot{\vec\eta}_i(\tau) = \frac{d\vec\eta_i(\tau)}{d\tau} = \frac{1}{c}\frac{d\vec\eta^{'}_i(t)}{dt}$ [25] and $\vec\beta^{(h)}_i = d^h\vec\beta_i/d\tau^h$ and writing the particle equations of motion as (no sum over i)

$$\frac{d}{d\tau}(m_i\frac{\vec\beta_i(\tau)}{\sqrt{1-\vec\beta^2_i(\tau)}}) = \frac{m_i}{\sqrt{1-\vec\beta^2_i(\tau)}}[\vec\beta^{(1)}_i + \vec\beta_i\frac{\vec\beta^{(1)}_i\cdot\vec\beta_i}{1-\vec\beta^2_i(\tau)}] \circeq Q_i\vec F_i, \qquad (5.14)$$

we obtain

$$m_i\frac{\vec\beta^{(1)}_i\cdot\vec\beta_i}{(1-\vec\beta^2_i(\tau))^{3/2}} \circeq Q_i\vec\beta_i\cdot\vec F_i, \qquad (5.15)$$

so that

$$\vec\beta^{(1)}_i \circeq \frac{\sqrt{1-\vec\beta^2_i(\tau)}}{m_i}Q_i(\vec F_i - \vec\beta_i\vec\beta_i\cdot\vec F_i). \qquad (5.16)$$

---

[25] $\tau = ct$, $\vec\eta^{'}_i(t) = \vec\eta_i(\tau)$; though using everywhere $c=1$, we have momentarily re-introduced it.

Thus in general we will have for every $h \geq 1$

$$\vec{\beta}_i^{(h)} \circeq Q_i \vec{G}_i, \tag{5.17}$$

so that using the Grassmann property of the charges

$$Q_i \vec{\beta}_i^{(h)} \circeq 0, \qquad h \geq 1. \tag{5.18}$$

This will lead to important simplifications later, allowing us to drop acceleration dependent terms in the force.

Due to the projector $P_\perp^{rs}(\vec{\sigma})$ required by the rest-frame radiation gauge, the sources of the transverse (Wigner spin 1) vector potential becomes non-local and one has a system of integrodifferential equations (like the equations generated by Fokker-Tetrode-type actions) for which it is not known how to define an initial value problem.

Let us end this Section with some comments on the equations of motion of the spin. Eq.(5.12) shows that, besides the standard term for scalar particles [20], the particle current contains also a dipole term $P_\perp^{rs}(\vec{\sigma}) \sum_{i=1}^N \frac{|\check{\vec{\kappa}}_i(\tau)|}{m_i} \Big(- \frac{(\vec{\bar{S}}_{i\xi} \times \vec{\partial})^s}{\sqrt{m_i^2 + \check{\vec{\kappa}}_i^2}} + \frac{\vec{\bar{S}}_{i\xi}(\tau) \times \check{\vec{\kappa}}_i(\tau)\ \check{\vec{\kappa}}_i(\tau) \cdot \vec{\partial}}{\sqrt{m_i^2 + \check{\vec{\kappa}}_i^2(\tau)} (|\check{\vec{\kappa}}_i(\tau)| + \sqrt{m_i^2 + \check{\vec{\kappa}}_i^2(\tau)})} \Big) \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau))$ in accord with the fact that the spinning particle has a pole-dipole structure [25] according to Papapetrou's classification [26] (see Refs. [27] for other pole-dipole models and Ref. [28] for their influence on the energy momentum tensor of action-at-a-distance models).

For N=1 ($\vec{\eta}_i \mapsto \vec{\eta}$, $m_i \mapsto m$,...) we have [26]:

i) $\bar{S}_\xi^{rs} = \epsilon^{rst} \bar{S}_\xi^t = -i \xi^r \xi^s$, $\bar{S}_s^{rs} \approx \int d^3\sigma [(\sigma^r - \eta^r)(\vec{\pi}_\perp \times \vec{B})^s - (\sigma^s - \eta^s)(\vec{\pi}_\perp \times \vec{B})^r](\tau, \vec{\sigma})$;

ii) $\Sigma_{s\mu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} p_s^\nu \tilde{S}_\xi^{\rho\sigma} = (\frac{1}{2} \epsilon_{\mu\nu ij} \delta^{ir} \delta^{js} - \epsilon_{\mu\nu oi} \frac{\delta^{ir} p_s^s}{p_s^o + \epsilon_s}) p_s^\nu \bar{S}_\xi^{rs}$.

The Bargmann-Michel-Telegdi equation [29] for the spin part of the Pauli-Lubanski 4-vector of Eq.(3.43)

$$\dot{\Sigma}_\mu \circeq \frac{e}{m} F_{\mu\nu} \Sigma^\nu, \tag{5.19}$$

for a spinning particle in an external electromagnetic field is replaced by the following equation in the canonical realization (3.29) of the *external* Poincaré group in the rest-frame instant form for the isolated system of a positive-energy spinning particle plus the electromagnetic field ($\dot{p}_s^\mu = 0$)

$$\begin{aligned} \dot{\Sigma}_{s\mu}(\tau) &= (\frac{1}{2} \epsilon_{\mu\nu ij} \delta^{ir} \delta^{js} - \epsilon_{\mu\nu oi} \frac{\delta^{ir} p_s^s}{p_s^o + \epsilon_s}) p_s^\nu \epsilon^{rst} \dot{\bar{S}}_\xi^t(\tau) \circeq \\ &\circeq (\frac{1}{2} \epsilon_{\mu\nu ij} \delta^{ir} \delta^{js} - \epsilon_{\mu\nu oi} \frac{\delta^{ir} p_s^s}{p_s^o + \epsilon_s}) p_s^\nu \epsilon^{rst} \Omega^{tn}(\tau) \bar{S}_\xi^n(\tau). \end{aligned} \tag{5.20}$$

In the last line we used Eqs.(5.7) for the spin.

---

[26] See Eq.(3.43) for the definition of $\Sigma_\mu = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} P^\nu S^{\rho\sigma}$ with $\dot{P}^\mu \neq 0$.

VI. THE SEMI-CLASSICAL LIENARD-WIECHERT SOLUTION, THE SECOND CLASS CONSTRAINTS AND THE REDUCED PHASE SPACE.

In this Section we study the retarded, advanced and symmetric Lienard-Wiechert solutions of Eqs.(5.12). After an equal time development of the delay, we show that, by using the particle equations of motion, the higher order accelerations disappear by virtue of $Q_i^2 = 0$. Therefore, at the semi-classical level there is a *unique* Lienard-Wiechert solution. Then we obtain the phase space expression of the semi-classical solution. This allows to eliminate the radiation field degrees of freedom by adding second class constraints and to arrive at a reduced phase space containing only particle degrees of freedom. The result is a canonical basis of this reduced phase space.

To simplify the notation from now on we shall denote the Dirac observables $\check{\vec{\kappa}}_i$, $\check{\vec{A}}_\perp$,... as $\vec{\kappa}_i$, $\vec{A}_\perp$,...

A. The Lienard-Wiechert Solutions, the Equal Time Expansion of the Delay and the Semi-Classical Approximation.

The symmetric solution of Eq.(5.12) is

$$
\begin{aligned}
A^r_{\perp S}(\tau,\vec{\sigma}) &= \frac{1}{2}[A^r_{\perp +} + A^r_{\perp -}](\tau,\vec{\sigma}) = \\
&= \frac{1}{2}\sum_{i=1}^N \frac{Q_i}{2\pi}\int d\tau_1 d^3\sigma_1 P^{rs}_\perp(\vec{\sigma})\, \mathbf{V}^s_i(\tau_1)[\theta(\tau-\tau_1)+\theta(\tau-\tau_1)] \\
&\quad \delta[(\tau-\tau_1)^2 - (\vec{\sigma}-\vec{\sigma}_1)^2]\delta^3(\vec{\sigma}_1 - \vec{\eta}_i(\tau_1)) = \\
&= \sum_{i=1}^N \frac{Q_i}{2\pi c}\int dt_1 P^{rs}_\perp(\vec{\sigma})\mathbf{V}^s_i(\tau_1)\delta[(t-t_1)^2 - \frac{1}{c^2}(\vec{\sigma}-\vec{\eta}_i(ct_1))^2] := \\
&:= \sum_{i=1}^N Q_i A^r_{\perp Si}(\tau,\vec{\sigma}),
\end{aligned}
\tag{6.1}
$$

in which we have put $\tau = ct$, $\vec{\beta}_i(\tau) = \dot{\vec{\eta}}_i(\tau) = \frac{1}{c}\frac{d\vec{\eta}^{\,\prime}_i(t)}{dt}$ and $\vec{A}_{\perp +} = \vec{A}_{\perp RET}$ ($\vec{A}_{\perp -} = \vec{A}_{\perp ADV}$) for the retarded (advanced) solution. The equation for t_1 is $c^2(t-t_1)^2 = (\vec{\sigma}-\vec{\eta}_i(ct_1))^2$ with the two solutions being

$$
\begin{aligned}
t_{i+}(\tau,\vec{\sigma}) &= \frac{1}{c}\tau_{i+}(\tau,\vec{\sigma}) = t - \frac{1}{c}r_{i+}(\tau_{i+}(\tau,\vec{\sigma}),\vec{\sigma}) = \frac{\tau}{c} - T_{i+}(\tau,\vec{\sigma}), \\
t_{i-}(\tau,\vec{\sigma}) &= \frac{1}{c}\tau_{i-}(\tau,\vec{\sigma}) = t + \frac{1}{c}r_{i-}(\tau_{i-}(\tau,\vec{\sigma}),\vec{\sigma}) = \frac{\tau}{c} + T_{i-}(\tau,\vec{\sigma}),
\end{aligned}
\tag{6.2}
$$

for the retarded and for the advanced case respectively. The light cone delta function is

$$
\begin{aligned}
&\delta[(\tau-\tau_1)^2 - (\vec{\sigma}-\vec{\eta}_i(\tau_1))^2] = \frac{1}{c^2}\delta[(t-t_1)^2 - \frac{1}{c^2}(\vec{\sigma}-\vec{\eta}_i(ct_1))^2] = \\
&= \frac{\delta[\tau_{1-} - \tau_{i+}(\tau,\vec{\sigma})]}{2|\tau-\tau_1 - \vec{\beta}_i(\tau_1)\cdot(\vec{\sigma}-\vec{\eta}_i(\tau_1))|} + \frac{\delta[\tau_{1-} - \tau_{i-}(\tau,\vec{\sigma})]}{2|\tau-\tau_1 - \vec{\beta}_i(\tau_1)\cdot(\vec{\sigma}-\vec{\eta}_i(\tau_1))|}.
\end{aligned}
\tag{6.3}
$$

The relative space location between the field point and the retarded or advanced particle position is

$$\vec{\sigma} - \vec{\eta}_i(\tau_{i\pm}(\tau,\vec{\sigma})) = \vec{r}_{i\pm}(\tau_{i\pm}(\tau,\vec{\sigma}),\vec{\sigma}) = r_{i\pm}(\tau_{i\pm}(\tau,\vec{\sigma}),\vec{\sigma})\hat{r}_{i\pm}(\tau_{i\pm}(\tau,\vec{\sigma}),\vec{\sigma}), \tag{6.4}$$

and its length is related to the time interval by

$$r_{i\pm}(\tau_{i\pm}(\tau,\vec{\sigma}),\vec{\sigma}) = |\,\vec{\sigma} - \vec{\eta}_i(\tau_{i\pm}(\tau,\vec{\sigma}))| = cT_{i\pm}(\tau,\vec{\sigma}) = |\tau - \tau_{i\pm}(\tau,\vec{\sigma})|,$$

$$\Rightarrow \quad \tau - \tau_{i\pm}(\tau,\vec{\sigma}) = \pm cT_{i\pm}(\tau,\vec{\sigma}) = \pm r_{i\pm}(\tau_{i\pm}(\tau,\vec{\sigma}),\vec{\sigma}). \tag{6.5}$$

The effective spatial interval is defined by

$$\rho_{i\pm}(\tau_{i\pm}(\tau,\vec{\sigma}),\vec{\sigma}) = r_{i\pm}(\tau_{i\pm}(\tau,\vec{\sigma}),\vec{\sigma})[1 \mp \vec{\beta}_i(\tau_{i\pm}(\tau,\vec{\sigma}))\cdot\hat{r}_{i\pm}(\tau_{i\pm}(\tau,\vec{\sigma}),\vec{\sigma})]. \tag{6.6}$$

In terms of these variables, the retarded, advanced and time symmetric solutions are

$$\begin{aligned}
A^r_{\perp\pm}(\tau,\vec{\sigma}) &= P^{rs}_\perp(\vec{\sigma})\sum_{i=1}^N \frac{Q_i}{4\pi}\ \mathbf{V}^s_i(\tau_{i\pm}(\tau,\vec{\sigma}))\frac{1}{\rho_{i\pm}(\tau_{i\pm}(\tau,\vec{\sigma}),\vec{\sigma})},\\
A^r_{\perp S}(\tau,\vec{\sigma}) &= \sum_{i=1}^N Q_i A^r_{\perp\, Si}(\tau,\vec{\sigma}) =\\
&= P^{rs}_\perp(\vec{\sigma})\sum_{i=1}^N \frac{Q_i}{8\pi}[\mathbf{V}^s_i(\tau_{i+}(\tau,\vec{\sigma}))\frac{1}{\rho_{i+}(\tau_{i+}(\tau,\vec{\sigma}),\vec{\sigma})}\\
&\quad + \mathbf{V}^s_i(\tau_{i-}(\tau,\vec{\sigma}))\frac{1}{\rho_{i-}(\tau_{i-}(\tau,\vec{\sigma}),\vec{\sigma})}].
\end{aligned} \tag{6.7}$$

We use the Smart-Wintner expansion [30–32]

$$\begin{aligned}
f(\tau_{i\pm}) &= f(\tau \mp cT_{i\pm}(\tau_{i\pm}((\tau,\vec{\sigma}),\vec{\sigma}))) = f(\tau - [\pm r_{i\pm}(\tau_{i\pm}(\tau,\vec{\sigma}),\vec{\sigma})]) =\\
&= f(\tau) + \sum_{k=1}^\infty \frac{(-)^k}{k!}\frac{d^{k-1}}{d\tau^{k-1}}\left[\frac{df(\tau)}{d\tau}(\pm r_i(\tau,\vec{\sigma}))^k\right] =\\
&= \sum_{k=0}^\infty \frac{(-)^k}{k!}\frac{d^k}{d\tau^k}\left[f(\tau)\,(\pm r_i(\tau,\vec{\sigma}))^k\,[1\mp\vec{\beta}_i(\tau)\cdot\hat{r}_i(\tau,\vec{\sigma})]\right],
\end{aligned} \tag{6.8}$$

where

$$\vec{r}_i(\tau,\vec{\sigma}) \;=\; r_i(\tau,\vec{\sigma})\hat{r}_i(\tau,\vec{\sigma}) = \vec{\sigma} - \vec{\eta}_i(\tau) = \vec{r}_{i\pm}(\tau_\pm(\tau,\vec{\sigma}),\vec{\sigma})|_{\tau_{i\pm}(\tau,\vec{\sigma})=\tau},$$

$$f(\tau_{i\pm}) \mapsto P^{rs}_\perp(\vec{\sigma})\mathbf{V}^s_i(\tau_{i\pm})\frac{1}{\rho_{i\pm}(\tau_{i\pm})} = P^{rs}_\perp(\vec{\sigma})\mathbf{V}^s_i(\tau_{i\pm})\frac{1}{r_{i\pm}(\tau_{i\pm})[1\mp\vec{\beta}_i(\tau_{i\pm})\cdot\hat{r}_{i\pm}(\tau,\vec{\sigma})]}. \tag{6.9}$$

The last line in Eq.(6.8) is identical to the previous one since $\frac{dr_i(\tau,\vec{\sigma})}{d\tau} = -\vec{\beta}_i(\tau)\cdot\hat{r}_i(\tau,\vec{\sigma})$. Hence we get

$$A^r_{\perp\pm}(\tau,\vec{\sigma}) = P^{rs}_\perp(\vec{\sigma})\sum_{i=1}^N\frac{Q_i}{4\pi}\sum_{k=0}^\infty\frac{(\mp)^k}{k!}\frac{d^k}{d\tau^k}[\mathbf{V}^s_i(\tau)r_i^{k-1}(\tau,\vec{\sigma})], \quad A^r_{\perp S} = \frac{1}{2}(A^r_{\perp +} + A^r_{\perp -}). \tag{6.10}$$

In order to evaluate the above derivatives we need the Leibnitz formula for the kth derivative of the product $f(\tau)g(\tau)$

$$\frac{d^k}{d\tau^k}(fg) = \sum_{m=0}^{k} \frac{k!}{m!(k-m)!} \frac{d^m f}{d\tau^m} \frac{d^{k-m} g}{d\tau^{k-m}}, \tag{6.11}$$

Thus

$$\sum_{k=0}^{\infty} \frac{(\mp)^k}{k!} \frac{d^k}{d\tau^k}[\mathbf{V}_i^s r_i^{k-1}] = \sum_{k=0}^{\infty} \frac{(\mp)^k}{k!} \sum_{m=0}^{k} \frac{k!}{m!(k-m)!} \frac{d^{k-m}\mathbf{V}_i^s}{d\tau^{k-m}} \frac{d^m r_i^{k-1}}{d\tau^m} =$$

$$\sum_{m=0}^{\infty} \sum_{k=m}^{\infty} \frac{(\mp)^k}{m!(k-m)!} \frac{d^{k-m}\mathbf{V}_i^s}{d\tau^{k-m}} \frac{d^m r_i^{k-1}}{d\tau^m} = \sum_{m=0}^{\infty} \sum_{h=0}^{\infty} \frac{(\mp)^{h+m}}{m!h!} \frac{d^h \mathbf{V}_i^s}{d\tau^h} \frac{d^m r_i^{h+m-1}}{d\tau^m}. \tag{6.12}$$

Using the notation $\mathbf{V}_i^{(h)s} = \frac{d^h \mathbf{V}_i^s}{d\tau^h}$, we obtain the following expression for the vector potential

$$A^r_{\perp\pm}(\tau, \vec{\sigma}) = P^{rs}_{\perp}(\vec{\sigma}) \sum_{i=1}^{N} \frac{Q_i}{4\pi} \sum_{h=0}^{\infty} \frac{(\mp)^h}{h!} \mathbf{V}_i^{(h)s}(\tau) \phi_{i\pm,h}(\tau, \vec{\sigma}), \tag{6.13}$$

in which

$$\phi_{i\pm,h}(\tau, \vec{\sigma}) = \sum_{m=0}^{\infty} \frac{(\mp)^m}{m!} \frac{d^m r_i^{h+m-1}(\tau, \vec{\sigma})}{d\tau^m} = \sum_{m=0}^{\infty} \frac{(\mp)^m}{m!} \frac{d^m}{d\tau^m} [\sqrt{(\vec{\sigma} - \vec{\eta}_i(\tau))^2}]^{m+h-1}. \tag{6.14}$$

In order to display the result of the evaluation of the derivative we use the formula

$$\frac{d^m}{d\tau^m} R(f(\tau)) = \sum_{n=0}^{m} \sum_{n_1 n_2 ..} \frac{m!}{n_1! n_2! ..} \frac{d^n R(f(\tau))}{df^n}|_{f=f(\tau)} (\frac{1}{1!} \frac{df(\tau)}{d\tau})^{n_1} (\frac{1}{2!} \frac{d^2 f(\tau)}{d\tau^2})^{n_2} ... \tag{6.15}$$

(with the summations restricted so that $\sum_r n_r = n,\ \sum_r r n_r = m$) to obtain

$$\phi_{i\pm,h}(\tau, \vec{\sigma}) = \sum_{m=0}^{\infty} \frac{(\mp)^m}{m!} \sum_{n=0}^{m} \sum_{n_1 n_2 ..} \frac{m!}{n_1! n_2! ..}$$
$$\frac{\partial^n r_i^{m+h-1}(\tau, \vec{\sigma})}{\partial \vec{r}_i^n} \circ \left(\frac{-\vec{\beta}_i(\tau)}{1!} \right)^{n_1} \left(\frac{-\vec{\beta}_i^{(1)}(\tau)}{2!} \right)^{n_2} ... \tag{6.16}$$

In this expression the symbol $\circ$ represents a scalar product between the tensors to the left and to the right with the summation $\sum_r n_r = n$ indicating how the indices would be matched. Changing the m summation index to $k = m - n$ we obtain

$$\phi_{i\pm,h}(\tau, \vec{\sigma}) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{n_1 n_2 ..} \frac{(\mp)^{k+n} (-)^{\Sigma_r n_r = n}}{n_1! n_2! ..}$$
$$\frac{\partial^n r_i^{k+n+h-1}(\tau, \vec{\sigma})}{\partial \vec{r}_i^n} \circ \left(\frac{\vec{\beta}_i(\tau)}{1!} \right)^{n_1} \left(\frac{\vec{\beta}_i^{(1)}(\tau)}{2!} \right)^{n_2} ... \tag{6.17}$$

(in this latter summation $\sum_r n_r = n,\ \sum_r r n_r = n + k$.)

Now we can take advantage of the Grassmann charges to significantly simplify the above multi-summations. As we have seen above, with a semi-classical Q_i there are *no accelerations* on shell $[Q_i \vec{\beta}_i^{(h)} \circeq 0$ from Eq.(5.18)] in the equations of motion of the particle 'i', since both the Coulomb potential and the Lienard-Wiechert Lorentz force on particle 'i' produced by the other particles, i.e. $Q_i[\vec{E}_\perp(\tau, \vec{\eta}_i(\tau)) + \vec{\beta}_i(\tau) \times \vec{B}(\tau, \vec{\eta}_i(\tau))]$, are proportional to Q_i. Therefore, the full set of Hamilton equations (5.5), (5.6) for both fields and particles imply that at the semi-classical level we have a natural *order reduction* of the final particle equation of motion in the Lienard-Wiechert sector (only second order differential equations). One effect of this truncation is the elimination of multi-particle forces; all the interactions will be *pairwise*, in both the Lagrangian and Hamiltonian formalisms. This was to be expected since the rest-frame instant form is an equal-time description of the N particle system: (acceleration-independent) 3-body,.. N-body forces appear as soon as we go to a description with no concept of equal time, like in the standard approach with N first class constraints [44].

Thus the only contributing indices are $n_2 = n_3 = .. = 0$, $n_1 = n$ and our expression for the transverse vector potentials simplify to

$$\begin{aligned}
A^r_{\perp\pm}(\tau, \vec{\sigma}) &\circeq P^{rs}_\perp(\vec{\sigma}) \sum_{i=1}^N \frac{Q_i}{4\pi} \mathbf{V}^s_i(\tau) \phi_{i\pm,0}(\tau, \vec{\sigma}) \circeq \\
\circeq &\quad P^{rs}_\perp(\vec{\sigma}) \sum_{i=1}^N \frac{Q_i}{4\pi} \mathbf{V}^s_i(\tau) \sum_{n=0}^\infty \frac{(\pm)^n}{n!} \frac{\partial^n r_i^{n-1}(\tau, \vec{\sigma})}{\partial \vec{r}_i^n} \circ \left(\frac{\vec{\beta}_i(\tau)}{1!} \right)^n, \\
A^r_{\perp S}(\tau, \vec{\sigma}) &= \frac{1}{2} (A^r_{\perp +} + A^r_{\perp -})(\tau, \vec{\sigma}).
\end{aligned} \tag{6.18}$$

Since $r_i = \sqrt{\vec{r}_i^2}$, we see that for odd $n = 2m+1$ we get

$$\frac{\partial^{2m+1}}{\partial \vec{r}_i^{2m+1}} (\sqrt{\vec{r}_i^2})^{2m} = \frac{\partial^{2m+1}}{\partial \vec{r}_i^{2m+1}} (\vec{r}_i^2)^m = 0, \tag{6.19}$$

and this implies the equality of the retarded, advanced and symmetric Lienard-Wiechert potentials on-shell

$$\begin{aligned}
A^r_{\perp S}(\tau, \vec{\sigma}) \circeq A^r_{\perp\pm}(\tau, \vec{\sigma}) \circeq &\ P^{rs}_\perp(\vec{\sigma}) \sum_{i=1, i\neq u}^N \frac{Q_i}{4\pi} V^s_i(\tau) \\
&\times \sum_{m=0}^\infty \frac{1}{(2m)!} \left(\vec{\beta}_i(\tau) \cdot \frac{\partial^{2m}}{\partial \vec{r}_i^{2m}} \right) r_i^{2m-1}(\tau, \vec{\sigma}).
\end{aligned} \tag{6.20}$$

Therefore, at the semi-classical level there is only one Lienard-Wiechert sector with a uniquely determined standard action-at-a-distance interaction.

B. The Phase Space Expression of the Semi-Classical Lienard-Wiechert Solution.

We use a tensor notation to write the transverse symmetric vector potential above as

$$\vec{A}_{\perp S}(\tau, \vec{\sigma}) \circeq \vec{\vec{P}}_\perp(\vec{\sigma}) \cdot \sum_{i=1}^N \frac{Q_i}{4\pi} \vec{\mathbf{V}}_i(\tau) \sum_{m=0}^\infty \frac{\dot{\vec{\eta}}_{i j_1}(\tau) .. \dot{\vec{\eta}}_{i j_{2m}}(\tau)}{(2m)!} \frac{\partial^{2m} |\vec{\sigma} - \vec{\eta}_i(\tau)|^{2m-1}}{\partial \sigma_{j_1} .. \partial \sigma_{j_{2m}}}. \tag{6.21}$$

Using the definition of the Coulomb projection operator

$$P_{\perp hk}(\vec{\sigma})F(\vec{\sigma}) = \delta_{hk}F(\vec{\sigma}) - \frac{1}{4\pi}\int d^3\sigma' \frac{\partial^2}{\partial\sigma_h\partial\sigma_k}\frac{1}{|\vec{\sigma}' - \vec{\sigma}|}F(\vec{\sigma}'), \tag{6.22}$$

and compactifying the notation still further we obtain

$$\begin{aligned}\vec{A}_{\perp S}(\tau,\vec{\sigma}) \;&\circeq \sum_{i=1}^{N}\frac{Q_i}{4\pi}\sum_{m=0}^{\infty}\frac{1}{(2m)!}\Big[\vec{\mathbf{V}}_i(\tau)(\dot{\vec{\eta}}_i(\tau)\cdot\vec{\partial}_\sigma)^{2m})\,|\vec{\sigma}-\vec{\eta}_i(\tau)|^{2m-1} - \\ &-\frac{1}{4\pi}\int d^3\sigma'[\vec{\partial}_\sigma(\vec{\mathbf{V}}_i(\tau)\cdot\vec{\partial}_\sigma)\frac{1}{|\vec{\sigma}'-\vec{\sigma}|}](\dot{\vec{\eta}}_i(\tau)\cdot\vec{\partial}_{\sigma'})^{2m}\,|\vec{\sigma}'-\vec{\eta}_i(\tau)|^{2m-1}\Big].\end{aligned} \tag{6.23}$$

Integration by parts and changing from $\frac{\partial}{\partial\vec{\sigma}'}$ to $\frac{\partial}{\partial\vec{\sigma}}$ and translation gives

$$\begin{aligned}\vec{A}_{\perp S}(\tau,\vec{\sigma}) \;&\circeq \sum_{i=1}^{N}\frac{Q_i}{4\pi}\sum_{m=0}^{\infty}\frac{1}{(2m)!}\Big[\vec{\mathbf{V}}_i(\tau)(\dot{\vec{\eta}}_i(\tau)\cdot\vec{\partial}_\sigma)^{2m})\,|\vec{\sigma}-\vec{\eta}_i(\tau)|^{2m-1} - \\ &-\frac{1}{4\pi}\int d^3\sigma'\Big(\vec{\partial}_\sigma(\vec{\mathbf{V}}_i(\tau)\cdot\vec{\partial}_\sigma)(\dot{\vec{\eta}}_i(\tau)\cdot\vec{\partial}_\sigma)^{2m}\,\frac{1}{|\vec{\sigma}'-(\vec{\sigma}-\vec{\eta}_i(\tau))|}\Big)\,{\sigma'}^{2m-1}\Big].\end{aligned} \tag{6.24}$$

We remind ourselves that, in addition to a convective part, the generalized current include magnetic and electric parts

$$\vec{\mathbf{V}}_i(\tau) = \dot{\vec{\eta}}_i(\tau) - \frac{i\sqrt{1-\dot{\vec{\eta}}_i^{\,2}(\tau)}\xi_i^s(\tau)\vec{\xi}_i(\tau)\cdot\vec{\partial}_\sigma}{m_i} + \frac{i\sqrt{1-\dot{\vec{\eta}}_i^{\,2}(\tau)}\dot{\vec{\eta}}_i(\tau)\cdot\vec{\xi}_i(\tau)\xi_i^s(\tau)\dot{\vec{\eta}}_i(\tau)\cdot\vec{\partial}_\sigma}{m_i(1+\sqrt{1-\dot{\vec{\eta}}_i^{\,2}(\tau)})}, \tag{6.25}$$

in which the derivatives act on $\vec{\sigma}$. Due to the delta function in Eq.(5.12), this derivative can be replaced with minus the derivative on $\vec{\eta}_i(\tau)$.

The integral in Eq.(6.24) is finite, and thus we can view it as the $\Lambda \to \infty$ limit of an integral with a cutoff Λ and take the derivatives out. The integral is thus of the form

$$-\frac{1}{4\pi}\vec{\partial}_\sigma(\vec{\mathbf{V}}_i(\tau)\cdot\vec{\partial}_\sigma)(\dot{\vec{\eta}}_i(\tau)\cdot\vec{\partial}_\sigma)^{2m}\int d^3\sigma'\frac{{\sigma'}^{2m-1}}{|\vec{\sigma}'-(\vec{\sigma}-\vec{\eta}_i(\tau))|}, \tag{6.26}$$

and

$$\begin{aligned}\frac{1}{4\pi}\int_\Lambda d^3\sigma'\frac{{\sigma'}^{2m-1}}{|\vec{\sigma}'-(\vec{\sigma}-\vec{\eta}_i)|} &= \frac{1}{2}\int_0^\Lambda d\sigma'{\sigma'}^{2m+1}\int_{-1}^{1}\frac{dz}{\sqrt{\vec{\sigma}'^2+(\vec{\sigma}-\vec{\eta}_i)^2-2\sigma'|\vec{\sigma}-\vec{\eta}_i|z}} \\ &= \frac{1}{2}\int_0^\Lambda d\sigma'{\sigma'}^{2m+1}\frac{-1}{\sigma'|\vec{\sigma}-\vec{\eta}_i|}\sqrt{\vec{\sigma}'^2+(\vec{\sigma}-\vec{\eta}_i)^2-2\sigma|\vec{\sigma}-\vec{\eta}_i|} \\ &= -\frac{1}{2|\vec{\sigma}-\vec{\eta}_i|}\int_0^\Lambda d\sigma'{\sigma'}^{2m}(|\vec{\sigma}'-|\vec{\sigma}-\vec{\eta}_i||-|\vec{\sigma}'+|\vec{\sigma}-\vec{\eta}_i||) \\ &= \frac{\Lambda^{2m+1}}{2m+1}-\frac{|\vec{\sigma}-\vec{\eta}_i|^{2m+1}}{(2m+1)(2m+2)}.\end{aligned} \tag{6.27}$$

Note that the Λ cutoff will get killed by the σ derivatives. Thus, we obtain

$$\vec{A}_{\perp S}(\tau,\vec{\sigma}) = \sum_{i=1}^{N}\frac{Q_i}{4\pi}\sum_{m=0}^{\infty}\Big[\frac{1}{(2m)!}\mathbf{\vec{V}}_i(\tau)(\dot{\vec{\eta}}_i(\tau)\cdot\vec{\partial}_\sigma)^{2m}\,|\vec{\sigma}-\vec{\eta}_i(\tau)|^{2m-1} - \\ -\frac{1}{(2m+2)!}\vec{\partial}_\sigma(\mathbf{\vec{V}}_i(\tau)\cdot\vec{\partial}_\sigma)(\dot{\vec{\eta}}_i(\tau)\cdot\vec{\partial}_\sigma)^{2m}\,|\vec{\sigma}-\vec{\eta}_i(\tau)|^{2m+1}\Big] := \\ := \sum_{i=1}^{N} Q_i\vec{A}_{\perp Si}(\vec{\sigma}-\vec{\eta}_i(\tau),\dot{\vec{\eta}}_i(\tau)) := \vec{A}_{\perp 1}(\vec{\sigma},\tau)+\vec{A}_{\perp 2}(\vec{\sigma},\tau)+\vec{A}_{\perp 3}(\vec{\sigma},\tau). \quad (6.28)$$

We have introduced the notations

$$\vec{A}_{\perp 1}(\tau,\vec{\sigma}) = \sum_{i=1}^{N}\frac{Q_i}{4\pi}\sum_{m=0}^{\infty}\frac{1}{(2m)!}\mathbf{\vec{V}}_i(\tau)(\dot{\vec{\eta}}_i(\tau)\cdot\vec{\partial}_\sigma)^{2m}\,|\vec{\sigma}-\vec{\eta}_i(\tau)|^{2m-1},$$

$$\vec{A}_{\perp 2}(\tau,\vec{\sigma}) = (-)\sum_{i=1}^{N}\frac{Q_i}{4\pi}\Big[(1+\frac{i\dot{\vec{\eta}}_i(\tau)\cdot\vec{\xi}_i(\tau)\vec{\xi}_i(\tau)\cdot\vec{\partial}_\sigma\sqrt{1-\dot{\vec{\eta}}_i^{\,2}(\tau)}}{m_i(1+\sqrt{1-\dot{\vec{\eta}}_i^{\,2}(\tau)})}\sum_{m=0}^{\infty}\Big[\frac{1}{(2m+2)!} \\ \vec{\partial}_\sigma(\dot{\vec{\eta}}_i(\tau)\cdot\vec{\partial}_\sigma)^{2m+1}\,|\vec{\sigma}-\vec{\eta}_i(\tau)|^{2m+1}\Big], \quad (6.29)$$

and

$$\vec{A}_{\perp 3}(\tau,\vec{\sigma}) = -\sum_{i=1}^{N}\frac{Q_i}{4\pi}\frac{i\vec{\xi}_i(\tau)\cdot\vec{\partial}_i\sqrt{1-\dot{\vec{\eta}}_i^{\,2}(\tau)}}{m_i}\sum_{m=0}^{\infty}[\frac{1}{(2m+2)!}\ \vec{\partial}_\sigma(\vec{\xi}_i(\tau)\cdot\vec{\partial}_\sigma) \\ (\dot{\vec{\eta}}_i(\tau)\cdot\vec{\partial}_\sigma)^{2m}\,|\vec{\sigma}-\vec{\eta}_i(\tau)|^{2m+1}]. \quad (6.30)$$

Note that since

$$(\vec{\xi}_i(\tau)\cdot\vec{\partial}_\sigma)(\vec{\xi}_i(\tau)\cdot\vec{\partial}_i)F(|\vec{\sigma}-\vec{\eta}_i(\tau)|) = 0, \quad (6.31)$$

the $\vec{A}_{\perp 3}(\tau,\vec{\sigma})$ term is zero.

Using the result that $(\dot{\vec{\eta}}_i\cdot\vec{\partial}_\eta)^{2m}|\vec{\eta}|^{2m-1} = [(2m-1)!!]^2\frac{1}{|\vec{\eta}|}[\dot{\vec{\eta}}_i^{\,2} - (\dot{\vec{\eta}}_i\cdot\frac{\vec{\eta}}{|\vec{\eta}|})^2]^m$ [$\vec{\partial}_\eta = \partial/\partial\vec{\eta}$], we get

$$\vec{A}_{\perp 1}(\tau,\vec{\sigma}) = \sum_{i=1}^{N}\frac{Q_i}{4\pi}\sum_{m=0}^{\infty}\Big[\frac{[(2m-1)!!]^2}{(2m)!|\vec{\sigma}-\vec{\eta}_i|}\mathbf{\vec{V}}_i(\tau)(\dot{\vec{\eta}}_i^{\,2} - (\dot{\vec{\eta}}_i\cdot\frac{\vec{\sigma}-\vec{\eta}_i}{|\vec{\sigma}-\vec{\eta}_i|})^2)^m\Big]. \quad (6.32)$$

We point out that all three currents contribute to $\vec{A}_{\perp 1}$, while only the convective and spin-electric parts contribute to $\vec{A}_{\perp 2}$. Using

$$\frac{[(2m-1)!!]^2}{(2m)!} = \frac{(2m)!}{(m!)^2 2^{2m}} = \frac{(m-1/2)!}{\sqrt{\pi}m!} = \frac{\sqrt{\pi}(-)^{m-1}(m-1/2)}{(1/2-m)!m!} = \\ = \frac{\sqrt{\pi}(-)^m}{(-1/2-m)!m!} = (-)^m\binom{-1/2}{m} \quad (6.33)$$

we find that

$$\vec{A}_{\perp 1}(\tau,\vec{\sigma}) = \sum_{i=1}^{N}\frac{Q_i}{4\pi}\mathbf{\vec{V}}_i(\tau)\frac{1}{|\vec{\sigma}-\vec{\eta}_i|}\sum_{m=0}^{\infty}(-)^m\binom{-1/2}{m}\left(\dot{\vec{\eta}}_i^{\,2} - (\dot{\vec{\eta}}_i\cdot\frac{\vec{\sigma}-\vec{\eta}_i}{|\vec{\sigma}-\vec{\eta}_i|})^2\right)^m = \\ = \sum_{i=1}^{N}\frac{Q_i}{4\pi}\mathbf{\vec{V}}_i(\tau)\frac{1}{|\vec{\sigma}-\vec{\eta}_i|}\frac{1}{\sqrt{1-\dot{\vec{\eta}}_i^{\,2}+(\dot{\vec{\eta}}_i\cdot\frac{\vec{\sigma}-\vec{\eta}_i}{|\vec{\sigma}-\vec{\eta}_i|})^2}}. \quad (6.34)$$

In terms of canonically conjugate variables this becomes

$$\vec{A}_{\perp 1}(\tau,\vec{\sigma}) = \sum_{i=1}^N \frac{Q_i}{4\pi}\Big[\vec{\kappa}_i - i\vec{\xi}_i(\tau)\vec{\xi}_i(\tau)\cdot\vec{\partial}_\sigma + \frac{i\vec{\kappa}_i(\tau)\cdot\vec{\partial}_\sigma\vec{\kappa}_i(\tau)\cdot\vec{\xi}_i(\tau)\vec{\xi}_i(\tau)}{\sqrt{m_i^2+\vec{\kappa}_i^2}(m_i+\sqrt{m_i^2+\vec{\kappa}_i^2})}\Big]$$
$$\frac{1}{|\vec{\sigma}-\vec{\eta}_i|}\frac{1}{\sqrt{m_i^2+(\vec{\kappa}_i\cdot\frac{\vec{\sigma}-\vec{\eta}_i}{|\vec{\sigma}-\vec{\eta}_i|})^2}}. \tag{6.35}$$

For $\vec{A}_{\perp 2}(\tau,\vec{\sigma})$, we see from Eq.(6.25) that we need an expression for $(\dot{\vec{\eta}}_i(\tau)\cdot\vec{\partial}_\eta)^{2m+1}|\vec{\eta}|^{2m+1}$ and $\vec{\xi}_i(\tau)\cdot\vec{\partial}_\eta(\dot{\vec{\eta}}_i(\tau)\cdot\vec{\partial}_\eta)^{2m}|\vec{\eta}|^{2m+1}$. One can show by an induction procedure that

$$(\dot{\vec{\eta}}_i(\tau)\cdot\vec{\partial}_\sigma)^{2m+1}|\vec{\sigma}-\vec{\eta}_i|^{2m+1} = [(2m+1)!!]^2\sum_{l=0}^m(-)^l\binom{m}{l}(\dot{\vec{\eta}}_i^2(\tau))^{m-l}\frac{(\dot{\vec{\eta}}_i(\tau)\cdot\frac{\vec{\sigma}-\vec{\eta}_i}{|\vec{\sigma}-\vec{\eta}_i|})^{2l+1}}{2l+1}, \tag{6.36}$$

so that we get

$$\vec{A}_{\perp 2}(\tau,\vec{\sigma}) = -)\sum_{i=1}^N\frac{Q_i}{4\pi}(1-\frac{i\dot{\vec{\eta}}_i(\tau)\cdot\vec{\xi}_i(\tau)\vec{\xi}_i(\tau)\cdot\vec{\partial}_\sigma\sqrt{1-\dot{\vec{\eta}}_i^2(\tau)}}{m_i(1+\sqrt{1-\dot{\vec{\eta}}_i^2(\tau)})})\vec{\partial}_\sigma$$
$$\sum_{m=0}^\infty\frac{[(2m+1)!!]^2}{(2m+2)!}\left(\dot{\vec{\eta}}_i^2(\tau)\right)^m\sum_{l=0}^m(-)^l\binom{m}{l}(\dot{\vec{\eta}}_i^2(\tau))^{-l}\frac{(\dot{\vec{\eta}}_i(\tau)\cdot\frac{\vec{\sigma}-\vec{\eta}_i}{|\vec{\sigma}-\vec{\eta}_i|})^{2l+1}}{2l+1}. \tag{6.37}$$

Since we have

$$\frac{[(2m+1)!!]^2}{(2m+2)!} = (-)^{m+1}\binom{-1/2}{m+1} \tag{6.38}$$

we obtain (where $\mathbf{I}$ is the unit dyad)

$$\vec{A}_{\perp 2}(\tau,\vec{\sigma}) = -\sum_{i=1}^N\frac{Q_i}{4\pi}\Big(1+\frac{i\vec{\kappa}_i(\tau)\cdot\vec{\xi}_i(\tau)\vec{\xi}_i(\tau)\cdot\vec{\partial}_\sigma}{\sqrt{m_i^2+\vec{\kappa}_i^2}(m_i+\sqrt{m_i^2+\vec{\kappa}_i^2})}\Big)\frac{\vec{\kappa}_i}{|\vec{\sigma}-\vec{\eta}_i|}\cdot\Big(\mathbf{I}-\frac{\vec{\sigma}-\vec{\eta}_i}{|\vec{\sigma}-\vec{\eta}_i|}\frac{\vec{\sigma}-\vec{\eta}_i}{|\vec{\sigma}-\vec{\eta}_i|}\Big)$$
$$\Big(\frac{\sqrt{m_i^2+\vec{\kappa}_i^2}}{\sqrt{m_i^2+(\vec{\kappa}_i\cdot\frac{\vec{\sigma}-\vec{\eta}_i}{|\vec{\sigma}-\vec{\eta}_i|})^2}}-1\Big)\frac{\sqrt{m_i^2+\vec{\kappa}_i^2}}{\vec{\kappa}_i^2-(\vec{\kappa}_i\cdot\frac{\vec{\sigma}-\vec{\eta}_i}{|\vec{\sigma}-\vec{\eta}_i|})^2}, \tag{6.39}$$

After combining some terms in $\vec{A}_{\perp 2}(\tau,\vec{\sigma})$, our *exact* result for the vector potential is

$$\vec{A}_{\perp S}(\tau,\vec{\sigma}) = \sum_{i=1}^N\frac{Q_i}{4\pi}\Big(\Big[\vec{\kappa}_i - i\vec{\xi}_i(\tau)\vec{\xi}_i(\tau)\cdot\vec{\partial}_\sigma + \frac{i\vec{\kappa}_i(\tau)\cdot\vec{\partial}_\sigma\vec{\kappa}_i(\tau)\cdot\vec{\xi}_i(\tau)\vec{\xi}_i(\tau)}{\sqrt{m_i^2+\vec{\kappa}_i^2}(m_i+\sqrt{m_i^2+\vec{\kappa}_i^2})}\Big]$$
$$\frac{1}{|\vec{\sigma}-\vec{\eta}_i|}\frac{1}{\sqrt{m_i^2+(\vec{\kappa}_i\cdot\frac{\vec{\sigma}-\vec{\eta}_i}{|\vec{\sigma}-\vec{\eta}_i|})^2}} - (1+\frac{i\vec{\kappa}_i(\tau)\cdot\vec{\xi}_i(\tau)\vec{\xi}_i(\tau)\cdot\vec{\partial}_\sigma}{\sqrt{m_i^2+\vec{\kappa}_i^2}(m_i+\sqrt{m_i^2+\vec{\kappa}_i^2})})\frac{\vec{\kappa}_i}{|\vec{\sigma}-\vec{\eta}_i|}$$
$$\cdot(\mathbf{I}-\frac{\vec{\sigma}-\vec{\eta}_i}{|\vec{\sigma}-\vec{\eta}_i|}\frac{\vec{\sigma}-\vec{\eta}_i}{|\vec{\sigma}-\vec{\eta}_i|})\frac{\sqrt{m_i^2+\vec{\kappa}_i^2}}{\sqrt{m_i^2+(\vec{\kappa}_i\cdot\frac{\vec{\sigma}-\vec{\eta}_i}{|\vec{\sigma}-\vec{\eta}_i|})^2}(\sqrt{m_i^2+\vec{\kappa}_i^2}+\sqrt{m_i^2+(\vec{\kappa}_i\cdot\frac{\vec{\sigma}-\vec{\eta}_i}{|\vec{\sigma}-\vec{\eta}_i|})^2})}\Big) =$$

$$= \sum_{i=1}^{N} \frac{Q_i}{4\pi}\Big(\vec{\kappa}_i \cdot (\mathbf{I} + \frac{\vec{\sigma}-\vec{\eta}_i}{|\vec{\sigma}-\vec{\eta}_i|}\frac{\vec{\sigma}-\vec{\eta}_i}{|\vec{\sigma}-\vec{\eta}_i|})\frac{\sqrt{m_i^2+\vec{\kappa}_i^2}}{|\vec{\sigma}-\vec{\eta}_i|\sqrt{m_i^2+(\vec{\kappa}_i\cdot\frac{\vec{\sigma}-\vec{\eta}_i}{|\vec{\sigma}-\vec{\eta}_i|})^2}} -$$

$$-i\vec{\xi}_i(\tau)\vec{\xi}_i(\tau)\cdot\vec{\partial}_\sigma \frac{1}{|\vec{\sigma}-\vec{\eta}_i|\sqrt{m_i^2+(\vec{\kappa}_i\cdot\frac{\vec{\sigma}-\vec{\eta}_i}{|\vec{\sigma}-\vec{\eta}_i|})^2}} +$$

$$+\frac{i\vec{\kappa}_i(\tau)\cdot\vec{\partial}_\sigma\vec{\kappa}_i(\tau)\cdot\vec{\xi}_i(\tau)\vec{\xi}_i(\tau)}{\sqrt{m_i^2+\vec{\kappa}_i^2}(\sqrt{m_i^2+\vec{\kappa}_i^2}+m_i)}\frac{1}{|\vec{\sigma}-\vec{\eta}_i|\sqrt{m_i^2+(\vec{\kappa}_i\cdot\frac{\vec{\sigma}-\vec{\eta}_i}{|\vec{\sigma}-\vec{\eta}_i|})^2}} -$$

$$-\frac{i\vec{\kappa}_i(\tau)\cdot\vec{\xi}_i(\tau)\vec{\xi}_i(\tau)\cdot\vec{\partial}_\sigma}{(m_i+\sqrt{m_i^2+\vec{\kappa}_i^2})}\vec{\kappa}_i\cdot(\mathbf{I} - \frac{\vec{\sigma}-\vec{\eta}_i}{|\vec{\sigma}-\vec{\eta}_i|}\frac{\vec{\sigma}-\vec{\eta}_i}{|\vec{\sigma}-\vec{\eta}_i|})\frac{1}{|\vec{\sigma}-\vec{\eta}_i|\sqrt{m_i^2+(\vec{\kappa}_i\cdot\frac{\vec{\sigma}-\vec{\eta}_i}{|\vec{\sigma}-\vec{\eta}_i|})^2}}\Big). \tag{6.40}$$

The corresponding $\vec{E}_{\perp S}(\tau,\vec{\sigma})$ and $\vec{B}_{\perp S}(\tau,\vec{\sigma})$ fields have the series forms

$$\vec{E}_{\perp S}(\tau,\vec{\sigma}) = -\sum_{i=1}^{N}\frac{Q_i}{4\pi}\sum_{m=0}^{\infty}\Big[\frac{1}{(2m)!}\vec{\mathbf{V}}_i(\tau)(\dot{\vec{\eta}}_i(\tau)\cdot\vec{\partial}_\sigma)^{2m+1}\,|\vec{\sigma}-\vec{\eta}_i(\tau)|^{2m-1} -$$
$$-\frac{1}{(2m+2)!}\vec{\partial}_\sigma(\vec{\mathbf{U}}_i(\tau)\cdot\vec{\partial}_\sigma)(\dot{\vec{\eta}}_i(\tau)\cdot\vec{\partial}_\sigma)^{2m+1}\,|\vec{\sigma}-\vec{\eta}_i(\tau)|^{2m+1}\Big], \tag{6.41}$$

where

$$\vec{\mathbf{U}}_i(\tau) = \dot{\vec{\eta}}_i(\tau) - \frac{i\sqrt{1-\dot{\vec{\eta}}_i^2(\tau)}\dot{\vec{\eta}}_i(\tau)\cdot\vec{\xi}_i(\tau)\vec{\xi}_i(\tau)\,\dot{\vec{\eta}}_i(\tau)\cdot\vec{\partial}_\sigma}{m_i(1+\sqrt{1-\dot{\vec{\eta}}_i^2(\tau)})}, \tag{6.42}$$

and

$$\vec{B}_S(\tau,\vec{\sigma}) = -\sum_{i=1}^{N}\frac{Q_i}{4\pi}\sum_{m=0}^{\infty}\frac{1}{(2m)!}\vec{\mathbf{V}}_i(\tau)\times\vec{\partial}_\sigma(\dot{\vec{\eta}}_i(\tau)\cdot\vec{\partial}_\sigma)^{2m}\,|\vec{\sigma}-\vec{\eta}_i(\tau)|^{2m-1}. \tag{6.43}$$

The corresponding *exact* forms in terms of the canonical variables are

$$\vec{B}_{\perp S}(\tau,\vec{\sigma}) = -\sum_{j=1}^{N}\frac{Q_j}{4\pi}\Big[\vec{\kappa}_j - i\vec{\xi}_j\vec{\xi}_j\cdot\vec{\partial}_\sigma + \frac{i\vec{\kappa}_j\cdot\vec{\partial}_\sigma\vec{\kappa}_j(\tau)\cdot\vec{\xi}_j(\tau)\vec{\xi}_j(\tau)}{\sqrt{m_j^2+\vec{\kappa}_j^2}(m_j+\sqrt{m_j^2+\vec{\kappa}_j^2})}\Big]$$
$$\vec{\partial}_\sigma\Big[\frac{1}{|\vec{\sigma}-\vec{\eta}_j|}\frac{1}{\sqrt{m_j^2+(\vec{\kappa}_j\cdot\frac{\vec{\sigma}-\vec{\eta}_j}{|\vec{\sigma}-\vec{\eta}_j|})^2}}\Big], \tag{6.44}$$

and

$$\vec{\pi}_{\perp S}(\tau,\vec{\sigma}) = \vec{E}_{\perp S}(\tau,\vec{\sigma}) = -\sum_{j=1}^{N}\frac{Q_j}{4\pi}\Big[(\mathbf{I}+\frac{i\vec{\kappa}_j(\tau)\cdot\vec{\xi}_j(\tau)\vec{\xi}_j(\tau)\vec{\partial}_\sigma}{\sqrt{m_j^2+\vec{\kappa}_j^2}(m_j+\sqrt{m_j^2+\vec{\kappa}_j^2})})\cdot\frac{\vec{\sigma}-\vec{\eta}_j}{|\vec{\sigma}-\vec{\eta}_j|^3}$$
$$(\frac{m_j^2\sqrt{m_j^2+\vec{\kappa}_j^2}}{[m_j^2+(\vec{\kappa}_j\cdot\frac{\vec{\sigma}-\vec{\eta}_j}{|\vec{\sigma}-\vec{\eta}_j|})^2]^{3/2}} - 1) -$$
$$-i\vec{\xi}_j\vec{\xi}_j\cdot\vec{\partial}_\sigma\vec{\kappa}_j\cdot\vec{\partial}_\sigma\frac{1}{|\vec{\sigma}-\vec{\eta}_j|}\frac{1}{\sqrt{m_j^2+(\vec{\kappa}_j\cdot\frac{\vec{\sigma}-\vec{\eta}_j}{|\vec{\sigma}-\vec{\eta}_j|})^2}}\Big]. \tag{6.45}$$

For the evaluation of the field energy we need $\vec{E}^2_{\perp S} + \vec{B}^2_S$, while for the field 3-momentum we need $\vec{E}_{\perp S} \times \vec{B}_S$. Since the exact forms cannot be integrated to closed forms, we need the series forms for the fields. From Appendix D we get [$\vec{\eta}_{ij}(\tau) = \eta_{ij}(\tau)\hat{\eta}_{ij}(\tau) = \vec{\eta}_i(\tau) - \vec{\eta}_j(\tau)$; $\vec{\partial}_{ij} = \partial/\partial\vec{\eta}_{ij}$]

$$
\begin{aligned}
U(\tau) \;&:=\; \frac{1}{2}\int d^3\sigma(\vec{E}^2_{\perp S} + \vec{B}^2_S)(\tau,\vec{\sigma}) \;:=\; \sum_{i<j}^{1..N} \frac{Q_iQ_j}{4\pi} h_1(\dot{\vec{\eta}}_i, \dot{\vec{\eta}}_j, \vec{\eta}_{ij}) = \\
&= \sum_{i<j}^{1..N} \frac{Q_iQ_j}{4\pi} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Big[\vec{\mathbf{V}}_i \cdot \vec{\mathbf{V}}_j \frac{(\dot{\vec{\eta}}_i \cdot \vec{\partial}_{ij})^{2m+1}(\dot{\vec{\eta}}_j \cdot \vec{\partial}_{ij})^{2n+1}\eta_{ij}^{2n+2m+1}}{(2n+2m+2)!} - \\
&- \frac{(\vec{\mathbf{V}}_i \cdot \vec{\partial}_{ij})(\vec{\mathbf{U}}_j \cdot \vec{\partial}_{ij})(\dot{\vec{\eta}}_i \cdot \vec{\partial}_{ij})^{2m+1}(\dot{\vec{\eta}}_j \cdot \vec{\partial}_{ij})^{2n+1}\eta^{2n+2m+3}}{(2n+2m+4)!} + \\
&\quad - \frac{(\vec{\mathbf{U}}_i \cdot \vec{\partial}_{ij})(\vec{\mathbf{V}}_j \cdot \vec{\partial}_{ij})(\dot{\vec{\eta}}_i \cdot \vec{\partial}_{ij})^{2m+1}(\dot{\vec{\eta}}_j \cdot \vec{\partial}_{ij})^{2n+1}\eta^{2n+2m+3}}{(2n+2m+4)!} + \\
&\quad + \frac{(\vec{\mathbf{U}}_i \cdot \vec{\partial}_{ij})(\vec{\mathbf{U}}_j \cdot \vec{\partial}_{ij})(\dot{\vec{\eta}}_i \cdot \vec{\partial}_{ij})^{2m+1}(\dot{\vec{\eta}}_j \cdot \vec{\partial}_{ij})^{2n+1}\eta^{2n+2m+3}}{(2n+2m+4)!} + \\
&+ \vec{\mathbf{V}}_i \cdot \vec{\mathbf{V}}_j \frac{(\dot{\vec{\eta}}_i \cdot \vec{\partial}_{ij})^{2m}(\dot{\vec{\eta}}_j \cdot \vec{\partial}_{ij})^{2n}\vec{\partial}^2_{ij}\eta_{ij}^{2n+2m+1}}{(2n+2m+2)!} - \\
&- \frac{(\vec{\mathbf{V}}_i \cdot \vec{\partial}_{ij})(\vec{\mathbf{V}}_j \cdot \vec{\partial}_{ij})(\dot{\vec{\eta}}_i \cdot \vec{\partial}_{ij})^{2m}(\dot{\vec{\eta}}_j \cdot \vec{\partial}_{ij})^{2n}\eta_{ij}^{2n+2m+1}}{(2n+2m+2)!}\Big],
\end{aligned}
\tag{6.46}
$$

and

$$
\begin{aligned}
\int d^3\sigma(\vec{E}_{\perp S} \times \vec{B}_S)(\tau,\vec{\sigma}) \;&:=\; \sum_{i<j}^{1..N} Q_iQ_j \vec{h}_1(\dot{\vec{\eta}}_i, \dot{\vec{\eta}}_j, \vec{\eta}_{ij}) = \\
= \sum_{i<j}^{1..N} \frac{Q_iQ_j}{4\pi}\Big[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} &\Big(\vec{\partial}_{ij}[\vec{\mathbf{V}}_i \cdot \vec{\mathbf{V}}_j \frac{(\dot{\vec{\eta}}_i \cdot \vec{\partial}_{ij})^{2m+1}(\dot{\vec{\eta}}_j \cdot \vec{\partial}_{ij})^{2n}\eta_{ij}^{2n+2m+1}}{(2n+2m+2)!} - \\
- \frac{(\vec{\mathbf{U}}_i \cdot \vec{\partial}_{ij})(\vec{\mathbf{V}}_j \cdot \vec{\partial}_{ij})(\dot{\vec{\eta}}_i \cdot \vec{\partial}_{ij})^{2m+1}(\dot{\vec{\eta}}_j \cdot \vec{\partial}_{ij})^{2n}\eta_{ij}^{2n+2m+3}}{(2n+2m+4)!}] \\
- \frac{\vec{\mathbf{V}}_j(\vec{\mathbf{V}}_i - \vec{\mathbf{U}}_i) \cdot \vec{\partial}_{ij}(\dot{\vec{\eta}}_i \cdot \vec{\partial}_{ij})^{2m+1}(\dot{\vec{\eta}}_j \cdot \vec{\partial}_{ij})^{2n}\eta_{ij}^{2n+2m+1}}{(2n+2m+2)!}\Big) \\
+ (i \longleftrightarrow j)\Big].
\end{aligned}
\tag{6.47}
$$

C. Final Dirac brackets and their Darboux basis

Till now we have worked in the reduced phase space of N positive-energy charged spinning particles plus the transverse electromagnetic field. This is a well defined isolated system with a global Darboux basis [$\vec{\eta}_i, \vec{\kappa}_i, \vec{\xi}_i, \vec{A}_{\perp}(\tau,\vec{\sigma}), \vec{\pi}_{\perp}(\tau,\vec{\sigma})$] and a well defined physical Hamiltonian, the invariant mass $M = \mathcal{P}^{\tau}_{(int)}$ of Eqs.(4.7), (5.1). All possible configurations of motion take place in this reduced phase space. The space of solutions of Hamilton's equations is a symplectic space, since there is a definition of Poisson brackets on the space of solutions. In Ref. [5] we found that we could select a subset of solutions of the equations of motion which is still a symplectic manifold. The method we used was to add by hand a set of second class constraints *compatible with the equations of motion*: this amounts to the selection of a symplectic submanifold of the symplectic manifold of solutions. We follow the same method here including the pseudo-classical spin variables $\vec{\xi}_i$.

As in the case of scalar particle, the Grassmann truncated semiclassical Lienard Wiechert solution $\vec{A}_{\perp S}$ given in Eq.(6.40) for the vector potential, with $\vec{\pi}_{\perp S} = \vec{E}_{\perp S} = -\frac{\partial}{\partial\tau}\vec{A}_{\perp S}$ for the canonical conjugate field momentum given by Eq.(6.45), provides us such a set of second class constraints by way of

$$\vec{\chi}_1(\tau,\vec{\sigma}) = \vec{A}_{\perp}(\tau,\vec{\sigma}) - \sum_{i=1}^{N} Q_i \vec{A}_{\perp Si}(\vec{\sigma} - \vec{\eta}_i(\tau), \vec{\kappa}_i(\tau), \vec{\xi}_i(\tau)) \approx 0,$$
$$\vec{\chi}_2(\tau,\vec{\sigma}) = \vec{\pi}_{\perp}(\tau,\vec{\sigma}) - \sum_{i=1}^{N} Q_i \vec{\pi}_{\perp Si}(\vec{\sigma} - \vec{\eta}_i(\tau), \vec{\kappa}_i(\tau), \vec{\xi}_i(\tau)) \approx 0. \tag{6.48}$$

These constraints allow us to eliminate the canonical degrees of freedom of the radiation field and to get the symmetric semi-classical Lienard-Wiechert reduced phase space, in which there are only particle degrees of freedom. In analogy to what occurred in Ref. [5] for spinless particle the independent variables $\vec{\eta}_i, \vec{\kappa}_i$ and now $\vec{\xi}_i$ as well will no longer be canonical when one imposes these constraints by way of modified Dirac brackets.

To determine the effects of these constraints in the construction of Dirac brackets we must compute the 6x6 matrix of brackets

$$\begin{pmatrix} \{\vec{\chi}_1, \vec{\chi}_1\} & \{\vec{\chi}_1, \vec{\chi}_2\} \\ \{\vec{\chi}_2, \vec{\chi}_1\} & \{\vec{\chi}_2, \vec{\chi}_2\} \end{pmatrix}. \tag{6.49}$$

It turns out that this matrix bracket is relatively simple, due to the Grassmann charges. Consider, for example the case of two particles. The particle or Lienard-Wiechert parts of the matrix bracket vanish since $Q_1^2 = 0 = Q_2^2$ and cross terms vanish because they involve Poisson brackets of particle one variables with particle two variables. Thus the only part of the 6x6 matrix bracket that contributes is from the field variables. It has the form

$$\{\vec{\chi}_1(\tau,\vec{\sigma}_1), \vec{\chi}_2(\tau,\vec{\sigma}_2)\} = (\mathbf{I} - \frac{\vec{\partial}\vec{\partial}}{\vec{\partial}^2})\delta^3(\vec{\sigma}_1 - \vec{\sigma}_2), \tag{6.50}$$

and since

$$\{\vec{\chi}_1, \vec{\chi}_1\} = 0 = \{\vec{\chi}_2, \vec{\chi}_2\}, \tag{6.51}$$

only the 3x3 off diagonal portion contributes.

In order to have a well defined Dirac bracket we need to use a modified form of the Poisson bracket in which the inverse of the matrix of constraint Poisson brackets is used. Calling this matrix C, we define $\tilde{C}^{-1}$ so that $C\tilde{C}^{-1} = (\mathbf{I} - \frac{\vec{\partial}\vec{\partial}}{\vec{\partial}^2})\delta^3(\vec{\sigma}_1 - \vec{\sigma}_2)$. But the transverse form of the delta function allows us to use the idempotent property of the projector to show that the inverse of C in this sense is just C itself. In that case for two functions $f(\vec{\kappa}_i, \vec{\eta}_i), g(\vec{\kappa}_i, \vec{\eta}_i)$ of the particle variables the Dirac bracket becomes

$$\begin{aligned}
\{f,g\}^* = \{f,g\} - & \\
& -Big[\int d^3\sigma \{f, -\sum_i Q_i \vec{A}_{\perp Si}(\vec{\sigma} - \vec{\eta}_i(\tau), \vec{\kappa}_i(\tau), \vec{\xi}_i(\tau))\} \cdot \\
& \cdot \{-\sum_j Q_j \vec{\pi}_{\perp Sj}(\vec{\sigma} - \vec{\eta}_j(\tau), \vec{\kappa}_j(\tau), \vec{\xi}_i(\tau)), g\} - \\
& -\{f, -\sum_j Q_j \vec{\pi}_{\perp Sj}(\vec{\sigma} - \vec{\eta}_j(\tau), \vec{\kappa}_j(\tau), \vec{\xi}_i(\tau))\} \cdot \\
& \cdot \{-\sum_i Q_i \vec{A}_{\perp Si}(\vec{\sigma} - \vec{\eta}_i(\tau), \vec{\kappa}_i(\tau), \vec{\xi}_i(\tau)), g\}\Big]. \qquad (6.52)
\end{aligned}$$

This is the bracket for the new reduced symplectic manifold containing only particles. To find the new canonical basis for the particles with respect to these Dirac brackets, we define the following scalar function

$$\mathcal{K} = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} Q_i Q_j \mathcal{K}_{ij}(\vec{\xi}_i, \vec{\xi}_j; \vec{\kappa}_i, \vec{\kappa}_j; \vec{\eta}_i - \vec{\eta}_j), \qquad (6.53)$$

in which there appear the following functions

$$\begin{aligned}
\mathcal{K}_{ij} = \int d^3\vec{\sigma} [& \vec{A}_{\perp Si}(\vec{\sigma} - \vec{\eta}_i, \vec{\kappa}_i, \vec{\xi}_i) \cdot \vec{\pi}_{\perp Sj}(\vec{\sigma} - \vec{\eta}_j, \vec{\kappa}_j, \vec{\xi}_j) \\
& -\vec{A}_{\perp Sj}(\vec{\sigma} - \vec{\eta}_j, \vec{\kappa}_j, \vec{\xi}_j) \cdot \vec{\pi}_{\perp Si}(\vec{\sigma} - \vec{\eta}_i, \vec{\kappa}_i, \vec{\xi}_i)] = \mathcal{K}_{ij}(\vec{\xi}_i, \vec{\xi}_j; \vec{\kappa}_i, \vec{\kappa}_j; \vec{\eta}_i - \vec{\eta}_j) = -\mathcal{K}_{ji}. \\
& \qquad (6.54)
\end{aligned}$$

In Ref. [5] we found that the old canonical variables $\vec{\eta}_i$ and $\vec{\kappa}_i$ need to be modified to

$$\begin{aligned}
\tilde{\vec{\eta}}_i &= \vec{\eta}_i + \frac{1}{2} \sum_{j \neq i} Q_i Q_j \vec{\partial}_{\kappa_i} \mathcal{K}_{ij} = \vec{\eta}_i + \sum_{j \neq i} \frac{1}{2} Q_i Q_j \{\vec{\eta}_i, \mathcal{K}_{ij}\}, \\
\tilde{\vec{\kappa}}_i &= \vec{\kappa}_i - \frac{1}{2} \sum_{j \neq i} Q_i Q_j \vec{\partial}_{\eta_i} \mathcal{K}_{ij} = \vec{\kappa}_i + \sum_{j \neq i} \frac{1}{2} Q_i Q_j \{\vec{\kappa}_i, \mathcal{K}_{ij}\}, \qquad (6.55)
\end{aligned}$$

so that they satisfy

$$\begin{aligned}
\{\tilde{\vec{\eta}}_k, \tilde{\vec{\eta}}_l\}^* &= 0, \\
\{\tilde{\vec{\kappa}}_k, \tilde{\vec{\kappa}}_l\}^* &= 0, \\
\{\tilde{\vec{\eta}}_i, \tilde{\vec{\kappa}}_j\}^* &= \mathbf{I}\delta_{ij}. \qquad (6.56)
\end{aligned}$$

In analogy to the above modifications we assume that

$$\tilde{\vec{\xi}}_i = \vec{\xi}_i + \frac{1}{2} \sum_{j \neq i} Q_i Q_j \{\vec{\xi}_i, \mathcal{K}_{ij}\}. \qquad (6.57)$$

We now show that this definition, together with the previous ones, are sufficient to guarantee that the new variables are canonical. To this end we let both

$$\begin{aligned}
\tilde{\vec{\mu}}_i &= \vec{\mu}_i + \frac{1}{2} \sum_{j \neq i} Q_i Q_j \{\vec{\mu}_i, \mathcal{K}_{ij}\}, \\
\tilde{\vec{\nu}}_i &= \vec{\nu}_i + \frac{1}{2} \sum_{j \neq i} Q_i Q_j \{\vec{\nu}_i, \mathcal{K}_{ij}\}, \qquad (6.58)
\end{aligned}$$

represent any of the new candidate canonical variables. Then we get

$$
\begin{aligned}
\{\widetilde{\vec{\mu}}_i, \widetilde{\vec{\nu}}_j\}^* = \{\widetilde{\vec{\mu}}_i, \widetilde{\vec{\nu}}_j\} & \\
& -\Big[\int d^3\sigma \{\widetilde{\vec{\mu}}_i, -\sum_k Q_k \vec{A}_{\perp Sk}(\vec{\sigma} - \vec{\eta}_k(\tau), \vec{\kappa}_k(\tau), \vec{\xi}_k(\tau))\} \cdot \\
& \cdot \{-\sum_l Q_l \vec{\pi}_{\perp Sl}(\vec{\sigma} - \vec{\eta}_l(\tau), \vec{\kappa}_l(\tau), \vec{\xi}_l(\tau)), \widetilde{\vec{\nu}}_j\} + \\
& +[\int d^3\sigma \{\widetilde{\vec{\mu}}_i, -\sum_k Q_k \vec{\pi}_{\perp Sk}(\vec{\sigma} - \vec{\eta}_k(\tau), \vec{\kappa}_k(\tau), \vec{\xi}_k(\tau))\} \cdot \\
& \cdot \{-\sum_l Q_l \vec{A}_{\perp Sl}(\vec{\sigma} - \vec{\eta}_l(\tau), \vec{\kappa}_l(\tau), \vec{\xi}_l(\tau)), \widetilde{\vec{\nu}}_j\}\Big].
\end{aligned}
\tag{6.59}
$$

Since we have

$$
\begin{aligned}
\{\widetilde{\vec{\mu}}_i, \widetilde{\vec{\nu}}_j\} = \{\vec{\mu}_i, \vec{\nu}_j\} + \frac{1}{2}\{\vec{\mu}_i, \sum_{k\neq j} Q_j Q_k \{\vec{\nu}_j, \mathcal{K}_{jk}\}\} - \frac{1}{2}\{\vec{\nu}_j, \sum_{k\neq i} Q_i Q_k \{\vec{\mu}_i, \mathcal{K}_{ik}\}\} + & \\
+\frac{1}{4}\{\sum_{k\neq i} Q_i Q_k \{\vec{\mu}_i, \mathcal{K}_{ik}\}, \sum_{l\neq j} Q_j Q_l \{\vec{\nu}_j, \mathcal{K}_{jl}\}\}, &
\end{aligned}
\tag{6.60}
$$

we see from the definition of $\mathcal{K}_{ij}$ that the last bracket would require that either $i = j, i = l$, or $j = l$. Due to the Grassman charges this forces the last bracket to be zero. For similar reasons we can replace the new canonical variables by the old ones in the last two lines of Eq.(6.59) and with the sums truncating so that

$$
\begin{aligned}
\{\widetilde{\vec{\mu}}_i, \widetilde{\vec{\nu}}_j\} = \{\vec{\mu}_i, \vec{\nu}_j\} + \frac{1}{2}\{\vec{\mu}_i, \sum_{k\neq j} Q_j Q_k \{\vec{\nu}_j, \mathcal{K}_{jk}\}\} - \frac{1}{2}\{\vec{\nu}_j, \sum_{k\neq i} Q_i Q_k \{\vec{\mu}_i, \mathcal{K}_{ik}\}\} & \\
-Q_i Q_j \int d^3\sigma \Big[\{\vec{\mu}_i, \vec{A}_{\perp Si}(\vec{\sigma} - \vec{\eta}_i, \vec{\kappa}_i, \vec{\xi}_i)\} \cdot \{\vec{\pi}_{\perp Sj}(\vec{\sigma} - \vec{\eta}_j, \vec{\kappa}_j, \vec{\xi}_j), \vec{\nu}_j\} & \\
-\{\vec{\mu}_i, \vec{\pi}_{\perp Si}(\vec{\sigma} - \vec{\eta}_i, \vec{\kappa}_i, \vec{\xi}_i)\} \cdot \{\vec{A}_{\perp Sj}(\vec{\sigma} - \vec{\eta}_j, \vec{\kappa}_j, \vec{\xi}_j), \vec{\nu}_j\}\Big]. &
\end{aligned}
\tag{6.61}
$$

Using the definition of the $\mathcal{K}_{jk}$, we find that the second and third terms on the right hand side become

$$
\begin{aligned}
& \frac{1}{2}\{\vec{\mu}_i, \sum_{k\neq j} Q_j Q_k \{\vec{\nu}_j, \mathcal{K}_{jk}\}\} - \frac{1}{2}\{\vec{\nu}_j, \sum_{k\neq i} Q_i Q_k \{\vec{\mu}_i, \mathcal{K}_{ik}\}\} = \\
= & \sum_{k\neq j} Q_i Q_k \frac{1}{2}\Big(\delta_{ij} \int d^3\sigma \Big[\{\vec{\mu}_i, \{\vec{\nu}_i, \vec{A}_{\perp \mathrm{Si}}(\vec{\sigma} - \vec{\eta}_i, \vec{\kappa}_i, \vec{\xi}_i)\}\} \cdot \vec{\pi}_{\perp k}(\vec{\sigma} - \vec{\eta}_k, \vec{\kappa}_k, \vec{\xi}_k) - \\
& -\{\vec{\mu}_i, \{\vec{\nu}_i, \vec{\pi}_{\perp \mathrm{Si}}(\vec{\sigma} - \vec{\eta}_i, \vec{\kappa}_i, \vec{\xi}_i)\}\} \cdot \vec{A}_{\perp k}(\vec{\sigma} - \vec{\eta}_k, \vec{\kappa}_k, \vec{\xi}_k)\Big] - \\
& -\delta_{ij} \int d^3\sigma \Big[\{\vec{\nu}_i, \{\vec{\mu}_i, \vec{A}_{\perp \mathrm{Si}}(\vec{\sigma} - \vec{\eta}_i, \vec{\kappa}_i, \vec{\xi}_i)\}\} \cdot \vec{\pi}_{\perp k}(\vec{\sigma} - \vec{\eta}_k, \vec{\kappa}_k, \vec{\xi}_k) - \\
& -\{\vec{\nu}_i, \{\vec{\mu}_i, \vec{\pi}_{\perp \mathrm{Si}}(\vec{\sigma} - \vec{\eta}_i, \vec{\kappa}_i, \vec{\xi}_i)\}\} \cdot \vec{A}_{\perp k}(\vec{\sigma} - \vec{\eta}_k, \vec{\kappa}_k, \vec{\xi}_k)\Big]\Big) + \\
& +Q_j Q_i \frac{1}{2}\Big(\int d^3\sigma \Big[\{\vec{\mu}_i, \vec{\pi}_{\perp i}(\vec{\sigma} - \vec{\eta}_k, \vec{\kappa}_k, \vec{\xi}_k)\} \cdot \{\vec{\nu}_j, \vec{A}_{\perp Sj}(\vec{\sigma} - \vec{\eta}_j, \vec{\kappa}_j, \vec{\xi}_j)\} - \\
& -\{\vec{\mu}_i, \vec{A}_{\perp i}(\vec{\sigma} - \vec{\eta}_i, \vec{\kappa}_i, \vec{\xi}_i)\} \cdot \{\vec{\nu}_j, \vec{\pi}_{\perp Sj}(\vec{\sigma} - \vec{\eta}_j, \vec{\kappa}_j, \vec{\xi}_j)\}\Big]\Big) - \\
& -Q_j Q_i \frac{1}{2}\Big(\int d^3\sigma \Big[\{\vec{\nu}_j, \vec{\pi}_{\perp j}(\vec{\sigma} - \vec{\eta}_j, \vec{\kappa}_j, \vec{\xi}_j)\} \cdot \{\vec{\mu}_i, \vec{A}_{\perp \mathrm{Si}}(\vec{\sigma} - \vec{\eta}_i, \vec{\kappa}_i, \vec{\xi}_i)\} - \\
& -\{\vec{\nu}_j, \vec{A}_{\perp j}(\vec{\sigma} - \vec{\eta}_j, \vec{\kappa}_j, \vec{\xi}_j)\} \cdot \{\vec{\mu}_i, \vec{\pi}_{\perp \mathrm{Si}}(\vec{\sigma} - \vec{\eta}_i, \vec{\kappa}_i, \vec{\xi}_i)\}\Big]\Big).
\end{aligned}
\tag{6.62}
$$

Combining the terms proportional to δ_{ij} by using the Jacobi identity and the fact that $\{\vec{\nu}_i, \vec{\mu}_i\}$ has a zero bracket with both $\vec{\pi}_{\perp \mathrm{Si}}$ and $\vec{A}_{\perp \mathrm{Si}}$, implies that these terms vanish, while the remaining terms combine to exactly cancel the last two lines of Eq.(6.61). Thus we have that

$$\{\widetilde{\vec{\mu}}_i, \widetilde{\vec{\nu}}_j\}^* = \{\vec{\mu}_i, \vec{\nu}_j\}, \tag{6.63}$$

and hence the new dynamical variables have the same Dirac brackets as the Poisson brackets of the old dynamical variables, that is, they are canonical. Thus, not only do we have same brackets as in Eq.(6.56) but also

$$\begin{aligned}
\{\widetilde{\vec{\xi}}_i, \widetilde{\vec{\xi}}_j\}^* &= \{\vec{\xi}_i, \vec{\xi}_j\} = -i\mathbf{I}\delta_{ij} \\
\{\widetilde{\vec{\eta}}_i, \widetilde{\vec{\xi}}_j\}^* &= \{\vec{\eta}_i, \vec{\xi}_j\} = 0 \\
\{\widetilde{\vec{\kappa}}_i, \widetilde{\vec{\xi}}_j\}^* &= \{\vec{\kappa}_i, \vec{\xi}_j\} = 0.
\end{aligned} \tag{6.64}$$

As in Ref. [5] the rest frame condition

$$\begin{aligned}
\vec{\mathcal{H}}_p = \vec{\mathcal{P}}_{(int)} &= \sum_{i-1}^{N} \vec{\kappa}_i + \int d^3\sigma [\vec{\pi}_\perp \times \vec{B}](\tau, \vec{\sigma}) = \\
&= \sum_{i-1}^{N} \vec{\kappa}_i + \sum_{i<j} Q_i Q_j \int d^3\sigma \Big[\vec{\pi}_{\perp Si}(\vec{\sigma} - \vec{\eta}_i, \vec{\kappa}_i) \times (\vec{\partial}_\sigma \times \vec{A}_{\perp Sj}(\vec{\sigma} - \vec{\eta}_j, \vec{\kappa}_j)) + \\
&\quad + \vec{\pi}_{\perp Sj}(\vec{\sigma} - \vec{\eta}_j, \vec{\kappa}_j) \times (\vec{\partial}_\sigma \times \vec{A}_{\perp Si}(\vec{\sigma} - \vec{\eta}_i, \vec{\kappa}_i)) \Big] \approx 0.
\end{aligned} \tag{6.65}$$

can be rewritten in these new canonical variables as

$$\vec{\mathcal{H}}_p = \vec{\mathcal{P}}_{(int)} = \widetilde{\vec{\kappa}}_+ = \sum_{i=1}^{N} \widetilde{\vec{\kappa}}_i = 0. \tag{6.66}$$

This is due to the fact that the steps of expanding the cross products, integrating by parts and using the transverse gauge condition make no additonal reference to the spin dependence of the fields.

The previous results allow to get $\vec{\mathcal{P}}_{(int)}$ in terms of the final canonical variables. For the internal angular momentum in terms of these variables we get

$$\begin{aligned}
\mathcal{J}^r_{(int)} = \varepsilon^{rst} \bar{S}^{st}_s &= \sum_{i=1}^{N} (\vec{\eta}_i(\tau) \times \vec{\kappa}_i(\tau))^r - \frac{i}{2} \sum_{i=1}^{N} (\vec{\xi}_i \times \vec{\xi}_i)^r + \int d^3\sigma \, (\vec{\sigma} \times \, [\vec{\pi}_\perp \times \vec{B}](\tau, \vec{\sigma}))^r = \\
&= \sum_{i=1}^{N} [(\widetilde{\vec{\eta}}_i - \vec{\alpha}_i) \times (\widetilde{\vec{\kappa}}_i - \vec{\beta}_i)]^r - \frac{i}{2} \sum_{i=1}^{N} [(\widetilde{\vec{\xi}}_i - \vec{\gamma}_i) \times (\widetilde{\vec{\xi}}_i - \vec{\gamma}_i)]^r + \\
&\quad + \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i}^{N} Q_i Q_j \int d^3\sigma \, (\vec{\sigma} \times \, [\vec{\pi}_{\perp Si} \times (\vec{\partial}_\sigma \times \vec{A}_{\perp Sj})] + i \leftrightarrow j).
\end{aligned} \tag{6.67}$$

In Ref. [5] we showed that $-\vec{\alpha}_i \times \widetilde{\vec{\kappa}}_i - \widetilde{\vec{\eta}}_i \times \vec{\beta}_i$ exactly compensates the last expression for the field portion ($\vec{\alpha}_i \times \vec{\beta}_i$ turned out to be zero due to Grassman truncation). The arguement we used depended explicitly on the fact that $\vec{\pi}_{\perp Si}, \vec{A}_{\perp Sj}$ have expected transformation properties. In the case here of spinning particles, we need in addition the terms $-\vec{\gamma}_i \times \tilde{\vec{\xi}}_i - \tilde{\vec{\xi}}_i \times \vec{\gamma}_i$. Using the forms for $\vec{\alpha}_i$, $\vec{\beta}_i$, and $\vec{\gamma}_i$ given in Eqs.(6.55), (6.57), together with the expression for $\mathcal{K}_{ij}$, and expanding the cross products in the integral, using the transverse nature of the field together with vanishing surface terms, we find that

$$
\begin{aligned}
\mathcal{J}^r_{(int)} &= \sum_{i=1}^N (\tilde{\vec{\eta}}_i \times \widetilde{\vec{\kappa}_i})^r - \frac{i}{2}\sum_{i=1}^N (\tilde{\vec{\xi}}_i \times \tilde{\vec{\xi}}_i)^r + \\
&+ \frac{1}{2}\sum_{i=1}^N \sum_{j\neq i}^N Q_i Q_j \Big[(\vec{\eta}_i \times \vec{\partial}_{\eta_i} + \vec{\kappa}_i \times \vec{\partial}_{\kappa_i} + \vec{\xi}_i \times \vec{\partial}_{\xi_i})^r \\
&\int d^3\sigma (\vec{A}_{\perp Si} \cdot \vec{\pi}_{\perp Sj} - \vec{A}_{\perp Sj} \cdot \vec{\pi}_{\perp Si}) \Big] - \\
&- \frac{1}{2}\sum_{i=1}^N \sum_{j\neq i}^N Q_i Q_j \int d^3\sigma \Big[\vec{A}_{\perp Si} \times \vec{\pi}_{\perp Sj} - \\
- \vec{\sigma} \times (\vec{\partial}_\sigma & {A^k}_{\perp Si}) \pi^k_{\perp Sj} + \vec{A}_{\perp Sj} \times \vec{\pi}_{\perp Si} + \vec{\sigma} \times ({A^k}_{\perp Si} \vec{\partial}_\sigma \pi^k_{\perp Sj}) \Big].
\end{aligned}
\tag{6.68}
$$

In order to see how the cancellation between the last two lines works, we first spell out the new vector and scalar dependence of the vector potential and its canonical momentum. Their general forms are [from Eq.(6.40) and Eq.(6.45)]

$$
\begin{aligned}
\vec{A}_{\perp Si} &= \frac{1}{4\pi\rho_i}\Big[\vec{\kappa}_i f_i(\kappa_i^2, \vec{\kappa}_i \cdot \hat{\rho}_i) + \hat{\rho}_i g_i(\kappa_i^2, \vec{\kappa}_i \cdot \hat{\rho}_i) + \vec{\kappa}_i \cdot \vec{\xi}_i \vec{\xi}_i \sum_n h_n(\rho_i) l_{ni}(\kappa_i^2, \vec{\kappa}_i \cdot \hat{\rho}_i) + \\
&+ \frac{\vec{\kappa}_i \vec{\kappa}_i \cdot \vec{\xi}_i \vec{\xi}_i \cdot \hat{\rho}_i}{\rho_i} j_i(\kappa_i^2, \vec{\kappa}_i \cdot \hat{\rho}_i) + \frac{\hat{\rho}_i \vec{\kappa}_i \cdot \vec{\xi}_i \vec{\xi}_i \cdot \hat{\rho}_i}{\rho_i} m_i(\kappa_i^2, \vec{\kappa}_i \cdot \hat{\rho}_i) \Big], \\
\vec{\pi}_{\perp Si} &= \frac{1}{4\pi\rho_i^2}\Big[\hat{\rho}_i c_i(\kappa_i^2, \vec{\kappa}_i \cdot \hat{\rho}_i) + \\
&+ \vec{\xi}_i \vec{\xi}_i \cdot \hat{\rho}_i \sum_n e_n(\rho_i) d_{ni}(\kappa_i^2, \vec{\kappa}_i \cdot \hat{\rho}_i) + \vec{\xi}_i \vec{\xi}_i \cdot \vec{\kappa}_i \sum_n a_n(\rho_i) b_{ni}(\kappa_i^2, \vec{\kappa}_i \cdot \hat{\rho}_i) \Big],
\end{aligned}
\tag{6.69}
$$

where

$$
\hat{\rho}_i := \frac{(\vec{\sigma} - \vec{\eta}_i)}{|\vec{\sigma} - \vec{\eta}_i|}; \quad \rho_i := |\vec{\sigma} - \vec{\eta}_i| := |\vec{\rho}_i|; \quad \vec{\partial}_\sigma \hat{\rho}_i = \frac{1}{\rho_i}(\mathbf{I} - \hat{\rho}_i \hat{\rho}_i).
\tag{6.70}
$$

Now we use

$$
\vec{\eta}_i \times \vec{\partial}_{\eta_i} = -\vec{\sigma} \times \vec{\partial}_\sigma + \vec{\rho}_i \times \vec{\partial}_\sigma,
\tag{6.71}
$$

and several facts (using generic functions h):

$$
\begin{aligned}
\vec{\rho}_i \times \vec{\partial}_\sigma h(\rho_i) &= 0, \\
(\vec{\rho}_i \times \vec{\partial}_\sigma + \vec{\kappa}_i \times \vec{\partial}_{\kappa_i}) h_i(\kappa_i^2, \vec{\kappa}_i \cdot \hat{\rho}_i) &= 0, \\
(\vec{\rho}_i \times \vec{\partial}_\sigma + \vec{\kappa}_i \times \vec{\partial}_{\kappa_i}) \vec{\kappa}_i \cdot &= \vec{\kappa}_i \times,
\end{aligned}
$$

$$
\begin{aligned}
(\vec{\rho}_i \times \vec{\partial}_\sigma + \vec{\kappa}_i \times \vec{\partial}_{\kappa_i})\hat{\rho}_i \cdot \; &= \hat{\rho}_i \times, \\
(\vec{\rho}_i \times \vec{\partial}_\sigma + \vec{\kappa}_i \times \vec{\partial}_{\kappa_i} + \vec{\xi}_i \times \vec{\partial}_{\xi_i})(\vec{\kappa}_i \cdot \vec{\xi}_i)\vec{\xi}_i \cdot \; &= \vec{\xi}_i \times, \\
(\vec{\rho}_i \times \vec{\partial}_\sigma + \vec{\kappa}_i \times \vec{\partial}_{\kappa_i} + \vec{\xi}_i \times \vec{\partial}_{\xi_i})\vec{\kappa}_i \cdot \vec{\xi}_i \vec{\xi}_i \cdot \hat{\rho}_i \vec{\kappa}_i \cdot \; &= \vec{\kappa}_i \cdot \vec{\xi}_i \vec{\xi}_i \cdot \hat{\rho}_i \vec{\kappa} \times, \\
(\vec{\rho}_i \times \vec{\partial}_\sigma + \vec{\kappa}_i \times \vec{\partial}_{\kappa_i} + \vec{\xi}_i \times \vec{\partial}_{\xi_i})\hat{\rho}_i \vec{\kappa}_i \cdot \vec{\xi}_i \vec{\xi}_i \cdot \hat{\rho}_i \hat{\rho}_i \cdot \; &= \hat{\rho}_i \vec{\kappa}_i \cdot \vec{\xi}_i \vec{\xi}_i \cdot \hat{\rho}_i \hat{\rho}_i \times, \\
(\vec{\rho}_i \times \vec{\partial}_\sigma + \vec{\kappa}_i \times \vec{\partial}_{\kappa_i} + \vec{\xi}_i \times \vec{\partial}_{\xi_i})(-)\vec{\xi}_i \vec{\xi}_i \cdot \hat{\rho}_i \vec{\xi}_i \cdot \; &= (-)\vec{\xi}_i \vec{\xi}_i \cdot \hat{\rho}_i \vec{\xi}_i \times . \qquad (6.72)
\end{aligned}
$$

where the $\times$ indicates vector cross product. Thus the effect of the bracket $(\vec{\rho}_i \times \vec{\partial}_\sigma + \vec{\kappa}_i \times \vec{\partial}_{\kappa_i} + \vec{\xi}_i \times \vec{\partial}_{\xi_i})$ is to turn scalar product into cross product, which together with the first part of Eq.(6.68) leads to the cancellation referred to above yielding

$$
\vec{\mathcal{J}}_{(int)} = \sum_{i=1}^{N} (\widetilde{\vec{\eta}_i} \times \widetilde{\vec{\kappa}_i}) - \frac{i}{2} \sum_{i=1}^{N} (\widetilde{\vec{\xi}_i} \times \widetilde{\vec{\xi}_i}). \qquad (6.73)
$$

Thus the total angular momentum of fields plus particles reduces to an expression involving just the new canonical particle variables when the fields are eliminated in pairs by using the modified Dirac brackets.

VII. THE SEMI-CLASSICAL HAMILTONIAN WITH THE DARWIN AND SPIN-DEPENDENT POTENTIALS.

In this Section we study the Hamiltonian M of Eq.(5.1), which contains both scalar (inside and outside the square roots) and vector (inside the square roots) direct interparticle potentials. By using the semi-classical property $Q_i^2 = 0$, these potentials can be reexpressed in terms of a unique scalar potential outside the square roots. This semi-classical potential contains the Coulomb potential and, moreover, semi-classical relativistic Darwin and spin-dependent potentials. The determination of these potentials will be done firstly by using the old variables (not canonical with respect to the Dirac brackets (6.52)) and then in the final canonical variables.

A. The Semi-Classical Hamiltonian in the Old Variables.

If we rewrite the Hamiltonian M (5.1) in the following form

$$\begin{aligned} M &= \sum_{i=1}^N \Big[\sqrt{m_i^2 + (\vec{\kappa}_i(\tau) - Q_i\ \vec{A}_\perp(\tau, \vec{\eta}_i(\tau)))^2} + \frac{\mathcal{R}_i}{2\sqrt{m_i^2 + \vec{\kappa}_i(\tau)^2}}\Big] + \\ &+ \sum_{i\neq j} \frac{Q_iQ_j}{4\pi \mid \vec{\eta}_i(\tau) - \vec{\eta}_j(\tau) \mid}\Big[1 - i\frac{\vec{\kappa}_i \cdot \vec{\xi}_i\vec{\xi}_i \cdot (\vec{\eta}_i(\tau) - \vec{\eta}_j(\tau))}{\mid \vec{\eta}_i(\tau) - \vec{\eta}_j(\tau) \mid^2 (m_i + \sqrt{m_i^2 + \vec{\kappa}_i^2})\sqrt{m_i^2 + \vec{\kappa}_i^2}}\Big] + \\ &+ \int d^3\sigma \frac{1}{2}[\vec{\pi}_\perp^2 + \vec{B}^2](\tau, \vec{\sigma}) = \\ &:= \sum_{i=1}^N \sqrt{m_i^2 + \vec{\kappa}_i(\tau)^2} + \sum_{i\neq j} \frac{Q_iQ_j}{4\pi \mid \vec{\eta}_i(\tau) - \vec{\eta}_j(\tau) \mid} + V_{DS}(\tau), \end{aligned} \tag{7.1}$$

with the scalar potential $V_{DS}(\tau)$ containing the generalized Darwin and spin-dependent interactions. This potential can be decomposed in three terms

$$V_{DS}(\tau) := U_1(\tau) + V_{DSO}(\tau) + U(\tau), \tag{7.2}$$

where $U(\tau)$, given in Eq.(6.46), contains the contribution coming from the radiation field energy evaluated on the semi-classical Lienard-Wiechert solution.

The potential $U_1(\tau)$ has the form

$$U_1(\tau) = \sum_{i=1}^N \frac{\mathcal{R}_i - 2Q_i\vec{\kappa}_i(\tau) \cdot \vec{A}_\perp(\tau, \vec{\eta}_i(\tau))}{2\sqrt{m_i^2 + \vec{\kappa}_i(\tau)^2}}, \tag{7.3}$$

containing the Grassman truncated vector potential, which was in the kinetic terms under the square roots, and the spin-magnetic field and spin-electric field interactions present in Eq.(5.2)

$$\begin{aligned} \mathcal{R}_i &= iQ_i\vec{\xi}_i \times \vec{\xi}_i \cdot \vec{B}(\tau, \vec{\eta}_i) - 2i\frac{Q_i\vec{\kappa}_i \cdot \vec{\xi}_i\vec{\xi}_i \cdot \vec{\pi}_\perp(\tau, \vec{\eta}_i)}{m_i + \sqrt{m_i^2 + \vec{\kappa}_i^2}} = \\ &= -2Q_i\vec{\bar{S}}_{i\xi}(\tau) \cdot \Big[\vec{B}(\tau, \vec{\eta}_i(\tau)) + \frac{\vec{\pi}_\perp(\tau, \vec{\eta}_i(\tau)) \times\ \vec{\kappa}_i(\tau)}{m_i + \sqrt{m_i^2 + \vec{\kappa}_i^2(\tau)}}\Big]. \end{aligned} \tag{7.4}$$

Finally, the potential $V_{DSO}(\tau)$ come from the interaction of the spin with the Coulomb electric field (after quantization it will contain important Darwin and spin orbit terms)

$$V_{DSO}(\tau) = -i \sum_{i\neq j} \frac{Q_iQ_j}{4\pi \mid \vec{\eta}_i(\tau) - \vec{\eta}_j(\tau) \mid^3} \frac{\vec{\kappa}_i \cdot \vec{\xi}_i\vec{\xi}_i \cdot (\vec{\eta}_i(\tau) - \vec{\eta}_j(\tau))}{(m_i + \sqrt{m_i^2 + \vec{\kappa}_i^2})\sqrt{m_i^2 + \vec{\kappa}_i^2}} =$$
$$= \sum_{i\neq j} \frac{Q_iQ_j}{4\pi \mid \vec{\eta}_i(\tau) - \vec{\eta}_j(\tau) \mid^3} \frac{\vec{\kappa}_i(\tau) \cdot (\vec{\eta}_i(\tau) - \vec{\eta}_j(\tau)) \times \vec{\bar{S}}_{i\xi}(\tau)}{(m_i + \sqrt{m_i^2 + \vec{\kappa}_i^2})\sqrt{m_i^2 + \vec{\kappa}_i^2}}. \qquad (7.5)$$

The rest-frame conditions are given by the vanishing of the *internal* 3-momentum [the last line is Eq.(6.66)]

$$\vec{\mathcal{P}}_{(int)} = \vec{\mathcal{H}}_p(\tau) = \vec{\kappa}_+(\tau) + \int d^3\sigma [\vec{\pi}_{\perp S} \times \vec{B}_S](\tau, \vec{\sigma}) = \sum_{i=1}^N \tilde{\vec{\kappa}}_+ \approx 0, \qquad (7.6)$$

with the term coming from radiation field 3-momentum given in Eq.(6.47).

Using the closed forms for the vector potential and the electric and magnetic fields given in Eqs. (6.40), (6.45) and (6.44), respectively, we can evaluate both the spin-independent and spin-dependent parts of the scalar potential $U_1(\tau)$ of Eq.(7.3).

While the last term in $U_1(\tau)$ gives

$$-\sum_{i=1}^N \frac{Q_i\vec{\kappa}_i(\tau) \cdot \vec{A}_\perp(\tau, \vec{\eta}_i(\tau))}{\sqrt{m_i^2 + \check{\vec{\kappa}}_i(\tau)^2}} =$$
$$= -\sum_{i\neq j}^N \frac{Q_jQ_i}{4\pi\sqrt{m_i^2 + \vec{\kappa}_i{}^2}} \Big(\Big[\vec{\kappa}_i + \frac{i\vec{\kappa}_j \cdot \vec{\xi}_j\vec{\kappa}_i \cdot \vec{\xi}_j\vec{\partial}_{ij}}{\sqrt{m_j^2 + \vec{\kappa}_j^2}(m_j + \sqrt{m_j^2 + \vec{\kappa}_j^2})} \Big] \cdot [\vec{\kappa}_j + \hat{\eta}_{ij}\vec{\kappa}_j \cdot \hat{\eta}_{ij}]$$
$$\frac{1}{|\vec{\eta}_i - \vec{\eta}_j|(\sqrt{m_j^2 + \vec{\kappa}_j^2} + \sqrt{m_j^2 + (\vec{\kappa}_j \cdot \hat{\eta}_{ij})^2})} - \vec{\kappa}_i \cdot \vec{\xi}_j\vec{\xi}_j \cdot \vec{\partial}_{ij} \frac{1}{|\vec{\eta}_i - \vec{\eta}_j| \sqrt{m_j^2 + (\vec{\kappa}_j \cdot \hat{\eta}_{ij})^2}} \Big), \qquad (7.7)$$

the magnetic and electric terms in $\mathcal{R}_i(\tau)$ give

$$+\sum_{i=1}^N \frac{iQ_i\vec{\xi}_i \times \vec{\xi}_i \cdot \check{\vec{B}}(\tau, \vec{\eta}_i)}{2\sqrt{m_i^2 + \check{\vec{\kappa}}_i(\tau)^2}} =$$
$$= -\sum_{i\neq j}^N \frac{iQ_iQ_j\vec{\xi}_i \times \vec{\xi}_i \cdot \{[\vec{\kappa}_j - i\vec{\xi}_j\vec{\xi}_j \cdot \vec{\partial}_{ij} + \frac{i\vec{\kappa}_j \cdot \vec{\partial}\vec{\kappa}_j(\tau) \cdot \vec{\xi}_j(\tau)\vec{\xi}_j(\tau)}{\sqrt{m_j^2 + \vec{\kappa}_j^2}(\sqrt{m_j^2 + \vec{\kappa}_j^2} + m_j)}]}{8\pi\sqrt{m_i^2 + \vec{\kappa}_i{}^2}}$$
$$\times \vec{\partial}_{ij} [\frac{1}{|\vec{\eta}_i - \vec{\eta}_j|} \frac{1}{\sqrt{m_j^2 + (\vec{\kappa}_j \cdot \hat{\eta}_{ij})^2}}]\}, \qquad (7.8)$$

and

$$-i\sum_{i=1}^N \frac{Q_i\check{\vec{\kappa}}_i \cdot \vec{\xi}_i\vec{\xi}_i \cdot \check{\vec{\pi}}_\perp(\tau, \vec{\eta}_i)}{\sqrt{m_i^2 + \check{\vec{\kappa}}_i{}^2}(m_i + \sqrt{m_i^2 + \check{\vec{\kappa}}_i^2})} =$$

$$= i\sum_{i\neq j}^{N}\frac{Q_iQ_j\vec{\kappa}_i\cdot\vec{\xi}_i}{\sqrt{m_i^2+\vec{\kappa_i}^2}(m_i+\sqrt{m_i^2+\vec{\kappa}_i^2})}\Big(\Big[\vec{\xi}_i+\frac{i\vec{\kappa}_j\cdot\vec{\xi}_j\vec{\xi}_i\cdot\vec{\xi}_j\vec{\partial}_{ij}}{\sqrt{m_j^2+\vec{\kappa}_j^2}(m_j+\sqrt{m_j^2+\vec{\kappa}_j^2})}\Big]\cdot\frac{\hat{\eta}_{ij}}{\eta_{ij}^2}$$
$$\Big(\frac{m_j^2\sqrt{m_j^2+\vec{\kappa}_j^2}}{[m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2]^{3/2}}-1\Big)-i\vec{\xi}_i\cdot\vec{\xi}_j\vec{\xi}_j\cdot\vec{\partial}_{ij}\vec{\kappa}_j\cdot\vec{\partial}_{ij}\frac{1}{\eta_{ij}}\frac{1}{\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2}}\Big), \tag{7.9}$$

respectively.

In Appendix D there is a comparison between these terms for the potential $U_1(\tau)$ with the terms appearing in the potential $U(\tau)$ of Eq.(6.47). Introducing the differential operators $\vec{\mathbf{k}}_i, \vec{\mathbf{l}}_i, \vec{\mathbf{f}}_i$ and $\vec{\mathbf{g}}_i$ defined by

$$\begin{aligned}
\vec{\mathbf{k}} &:= \vec{\kappa}_i+\vec{\mathbf{f}}_i, \quad \vec{\mathbf{f}}_i := -i\vec{\xi}_i\vec{\xi}_i\cdot\vec{\partial}_{ij}+\frac{i\vec{\kappa}_i\cdot\vec{\partial}_{ij}\vec{\kappa}_i\cdot\vec{\xi}_i\vec{\xi}_i}{(m_i+\sqrt{m_i^2+\vec{\kappa_i}^2})\sqrt{m_i^2+\vec{\kappa_i}^2}},\\
\vec{\mathbf{l}}_i &:= \vec{\kappa}_i+\vec{\mathbf{g}}_i, \quad \vec{\mathbf{g}}_i := \frac{i\vec{\kappa}_i\cdot\vec{\partial}_{ij}\vec{\kappa}_i\cdot\vec{\xi}_i\vec{\xi}_i}{(m_i+\sqrt{m_i^2+\vec{\kappa_i}^2})\sqrt{m_i^2+\vec{\kappa_i}^2}},
\end{aligned} \tag{7.10}$$

and combining all terms, we find that the potential $V_{DS}(\tau)$ can be rewritten in the form

$$V_{DS}(\tau) = V_{DSO}(\tau)+U_1(\tau)+U(\tau) = V_{LDS}+U_{HDS}, \tag{7.11}$$

where

$$\begin{aligned}
V_{LDS} &:= \sum_{i\neq j}\frac{Q_iQ_j}{4\pi}\Big[-\frac{\vec{\kappa}_i\cdot\vec{\kappa}_j+\vec{\kappa}_i\cdot\hat{\eta}_{ij}\vec{\kappa}_j\cdot\hat{\eta}_{ij}}{4\sqrt{m_i^2+\vec{\kappa_i}^2}\sqrt{m_j^2+\vec{\kappa_j}^2}}\frac{1}{\eta_{ij}}-i\frac{\vec{\kappa}_i\cdot\vec{\xi}_i\vec{\xi}_i\cdot\vec{\eta}_{ij}}{\eta_{ij}^3(m_i+\sqrt{m_i^2+\vec{\kappa}_i^2})\sqrt{m_i^2+\vec{\kappa}_i^2}}-\\
&-i\frac{\vec{\xi}_i\times\vec{\xi}_i\cdot(\vec{\mathbf{k}}_j\times\vec{\partial}_{ij})}{2\sqrt{m_i^2+\vec{\kappa_i}^2}}\frac{1}{\eta_{ij}\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2}}+i\frac{\vec{\kappa}_i\cdot\vec{\xi}_i}{\sqrt{m_i^2+\vec{\kappa_i}^2}(m_i+\sqrt{m_i^2+\vec{\kappa}_i^2})}\\
&\Big(\Big[\vec{\xi}_i+\frac{i\vec{\kappa}_j\cdot\vec{\xi}_j\vec{\xi}_i\cdot\vec{\xi}_j\vec{\partial}_{ij}}{\sqrt{m_j^2+\vec{\kappa}_j^2}(\sqrt{m_j^2+\vec{\kappa}_j^2}+m_j)}\Big]\cdot\frac{\hat{\eta}_{ij}}{\eta_{ij}^2}\Big(\frac{m_j^2\sqrt{m_j^2+\vec{\kappa}_j^2}}{[m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2]^{3/2}}-1\Big)-\\
&-i\vec{\xi}_i\cdot\vec{\xi}_j\vec{\xi}_j\cdot\vec{\partial}_{ij}\vec{\kappa}_j\cdot\vec{\partial}_{ij}\frac{1}{\eta_{ij}}\frac{1}{\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2}}\Big)+\\
&+\frac{\vec{\mathbf{f}}_{\mathbf{i}}\cdot(\vec{\kappa}_{\mathbf{j}}+\vec{\mathbf{f}}_{\mathbf{j}})}{\sqrt{m_j^2+\vec{\kappa_j}^2}}\frac{1}{\eta_{ij}\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2}}+\frac{\vec{\mathbf{f}}_{\mathbf{j}}\cdot(\vec{\kappa}_{\mathbf{i}}+\vec{\mathbf{f}}_{\mathbf{i}})}{\sqrt{m_i^2+\vec{\kappa_i}^2}}\frac{1}{\eta_{ij}\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2}}-\\
&-\frac{\vec{\kappa}_i\cdot\vec{\mathbf{f}}_j+\vec{\kappa}_j\cdot\vec{\mathbf{f}}_i+\vec{\mathbf{f}}_i\cdot\vec{\mathbf{f}}_j}{2\sqrt{m_i^2+\vec{\kappa_i}^2}\sqrt{m_j^2+\vec{\kappa_j}^2}}\frac{1}{\eta_{ij}}+\\
&+\Big(\vec{\kappa}_i\cdot\vec{\mathbf{f}}_j+\vec{\kappa}_j\cdot\vec{\mathbf{f}}_i+\vec{\mathbf{f}}_i\cdot\vec{\mathbf{f}}_j-(\vec{\mathbf{f}}_j\vec{\kappa}_i\cdot\hat{\eta}_{ij}+\vec{\mathbf{f}}_i\vec{\kappa}_j\cdot\hat{\eta}_{ij}+\vec{\mathbf{f}}_j\vec{\mathbf{f}}_i\cdot\hat{\eta}_{ij})\cdot\hat{\eta}_{ij}\Big)\\
&\Big(\frac{1}{2\sqrt{m_i^2+\vec{\kappa_i}^2}\sqrt{m_j^2+\vec{\kappa_j}^2}\eta_{ij}}-\frac{2}{\eta_{ij}\sqrt{m_j^2+\vec{\kappa_j}^2}(\sqrt{m_i^2+\vec{\kappa_i}^2}+\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2})}\Big)-\\
&-2\Big((\vec{\kappa}_i\cdot\vec{\kappa}_i)(\vec{\mathbf{f}}_j\cdot\vec{\kappa}_i-(\vec{\mathbf{f}}_j\cdot\hat{\eta}_{ij})(\vec{\kappa}_i\cdot\hat{\eta}_{ij}))-\vec{\mathbf{f}}_j\cdot[(\vec{\kappa}_i\cdot\hat{\eta}_{ij})(\vec{\kappa}_i-\hat{\eta}_{ij}(\vec{\kappa}_i\cdot\hat{\eta}_{ij}))](\vec{\kappa}_i\cdot\hat{\eta}_{ij})+\\
&+(\vec{\mathbf{f}}_i\cdot\vec{\kappa}_i)(\vec{\kappa}_j\cdot\vec{\kappa}_i-(\vec{\kappa}_j\cdot\hat{\eta}_{ij})(\vec{\kappa}_i\cdot\hat{\eta}_{ij}))-\vec{\kappa}_j\cdot[(\vec{\mathbf{f}}_i\cdot\hat{\eta}_{ij})(\vec{\kappa}_i-\hat{\eta}_{ij}(\vec{\kappa}_i\cdot\hat{\eta}_{ij}))](\vec{\kappa}_i\cdot\hat{\eta}_{ij})+
\end{aligned}$$

$$+(\vec{\mathbf{f}}_i\cdot\vec{\kappa}_i)(\vec{\mathbf{f}}_j\cdot\vec{\kappa}_i-(\vec{\mathbf{f}}_j\cdot\hat{\eta}_{ij})(\vec{\kappa}_i\cdot\hat{\eta}_{ij}))-\vec{\mathbf{f}}_j\cdot[(\vec{\mathbf{f}}_i\cdot\hat{\eta}_{ij})(\vec{\kappa}_i-\hat{\eta}_{ij}(\vec{\kappa}_i\cdot\hat{\eta}_{ij}))](\vec{\kappa}_i\cdot\hat{\eta}_{ij})\Big)$$

$$\frac{1}{\eta_{ij}\sqrt{m_j^2+\vec{\kappa}_j{}^2}\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2}}\frac{1}{(\sqrt{m_i^2+\vec{\kappa}_i{}^2}+\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2})^2})+$$

$$+[\vec{\kappa}_i\cdot\vec{\mathbf{g}}_j-\vec{\kappa}_i\cdot(\vec{\mathbf{g}}_j\cdot(\hat{\eta}_{ij}\hat{\eta}_{ij}))]$$

$$\frac{2}{\eta_{ij}\sqrt{m_i^2+\vec{\kappa}_i^2}\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2}}(\frac{\sqrt{m_j^2+\vec{\kappa}_j^2}}{\sqrt{m_j^2+\vec{\kappa}_j^2}+\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2}})\Big]. \tag{7.12}$$

The term V_{LDS} above includes all of the terms in Appendix D that are in closed form and are of order $1/c^2$ and higher while the term U_{HDS} below includes all of the terms that involve the double infinite series and are of order $1/c^4$ and higher.

$$U_{HDS}:=\sum_{i\neq j}\frac{Q_iQ_j}{8\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Big[\frac{\vec{\mathbf{k}}_{\mathbf{i}}}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}_i\cdot\frac{\vec{\mathbf{k}}_{\mathbf{j}}}{\sqrt{m_j^2+\vec{\kappa}_j{}^2}}$$

$$(\frac{\vec{\kappa}_i}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\cdot\vec{\partial}_{ij})^{2m+1}(\frac{\vec{\kappa}_j}{\sqrt{m_j^2+\vec{\kappa}_j{}^2}}\cdot\vec{\partial}_{ij})^{2n+1}\frac{\eta_{ij}^{2n+2m+1}}{(2n+2m+2)!}-$$

$$-(\frac{\vec{\mathbf{k}}_{\mathbf{i}}}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\cdot\vec{\partial}_{ij})(\frac{\vec{\mathbf{l}}_{\mathbf{j}}}{\sqrt{m_j^2+\vec{\kappa}_j{}^2}}\cdot\vec{\partial}_{ij})(\frac{\vec{\kappa}_i}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\cdot\vec{\partial}_{ij})^{2m+1}$$

$$(\frac{\vec{\kappa}_j}{\sqrt{m_j^2+\vec{\kappa}_j{}^2}}\cdot\vec{\partial}_{ij})^{2n+1}\frac{\eta_{ij}^{2n+2m+3}}{(2n+2m+4)!}-$$

$$-(\frac{\vec{\mathbf{l}}_{\mathbf{i}}}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\cdot\vec{\partial}_{ij})(\frac{\vec{\mathbf{k}}_{\mathbf{j}}}{\sqrt{m_j^2+\vec{\kappa}_j{}^2}}\cdot\vec{\partial}_{ij})(\frac{\vec{\kappa}_i}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\cdot\vec{\partial}_{ij})^{2m+1}$$

$$(\frac{\vec{\kappa}_j}{\sqrt{m_j^2+\vec{\kappa}_j{}^2}}\cdot\vec{\partial}_{ij})^{2n+1}\frac{\eta_{ij}^{2n+2m+3}}{(2n+2m+4)!}+$$

$$+(\frac{\vec{\mathbf{l}}_{\mathbf{i}}}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\cdot\vec{\partial}_{ij})(\frac{\vec{\mathbf{l}}_{\mathbf{j}}}{\sqrt{m_j^2+\vec{\kappa}_j{}^2}}\cdot\vec{\partial}_{ij})(\frac{\vec{\kappa}_i}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\cdot\vec{\partial}_{ij})^{2m+1}$$

$$(\frac{\vec{\kappa}_j}{\sqrt{m_j^2+\vec{\kappa}_j{}^2}}\cdot\vec{\partial}_{ij})^{2n+1}\frac{\eta_{ij}^{2n+2m+3}}{(2n+2m+4)!}+$$

$$+\frac{\vec{\mathbf{k}}_{\mathbf{i}}}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\cdot\frac{\vec{\mathbf{k}}_{\mathbf{j}}}{\sqrt{m_j^2+\vec{\kappa}_j{}^2}}(\frac{\vec{\kappa}_i}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\cdot\vec{\partial}_{ij})^{2m+2}(\frac{\vec{\kappa}_j}{\sqrt{m_j^2+\vec{\kappa}_j{}^2}}\cdot\vec{\partial}_{ij})^{2n+2}\frac{\eta_{ij}^{2n+2m+3}}{(2n+2m+4)!}-$$

$$-(\frac{\vec{\mathbf{k}}_{\mathbf{i}}}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\cdot\vec{\partial}_{ij})(\frac{\vec{\mathbf{k}}_{\mathbf{j}}}{\sqrt{m_j^2+\vec{\kappa}_j{}^2}}\cdot\vec{\partial}_{ij})(\frac{\vec{\kappa}_i}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\cdot\vec{\partial}_{ij})^{2m+2}$$

$$(\frac{\vec{\kappa}_j}{\sqrt{m_j^2+\vec{\kappa}_j{}^2}}\cdot\vec{\partial}_{ij})^{2n+2}\frac{\eta_{ij}^{2n+2m+5}}{(2n+2m+6)!}\Big]. \tag{7.13}$$

As discussed in Appendix D this complex formula for the lower and higher order Darwin and spin-dependent terms has been checked to reduce to the corresponding spinless results given in Eq.(6.14) in Ref. [5] when $\vec{\mathbf{f}}_i = \vec{\mathbf{g}}_i = 0$.

B. The Semi-Classical Hamiltonian in the Final Canonical Variables.

We now reexpress our final result in terms of the final canonical variables (6.56), (6.64) forming a Darboux basis for the Dirac brackets. These are the *physical* particle variable in this reduced phase space. By using Eqs.(6.55) we get

$$\sqrt{m_i^2 + \vec{\kappa}_i^2} = \sqrt{m_i^2 + \widetilde{\vec{\kappa}}_i^2 + \widetilde{\vec{\kappa}}_i \cdot \sum_{j\neq i} Q_i Q_j \vec{\partial}_{\eta_i} \mathcal{K}_{ij}} = $$
$$= \sqrt{m_i^2 + \widetilde{\vec{\kappa}}_i^2} + \frac{\widetilde{\vec{\kappa}}_i \cdot \sum_{j\neq i} Q_i Q_j \vec{\partial}_{\tilde{\eta}_i} \widetilde{\mathcal{K}}_{ij}}{2\sqrt{m_i^2 + \widetilde{\vec{\kappa}}_i^2}}, \qquad (7.14)$$

with

$$Q_i Q_j \mathcal{K}_{ij}(\vec{\kappa}_i, \vec{\kappa}_j, \vec{\eta}_i - \vec{\eta}_j; \vec{\xi}_i, \vec{\xi}_j) = Q_i Q_j \widetilde{\mathcal{K}}_{ij}(\widetilde{\vec{\kappa}}_i, \widetilde{\vec{\kappa}}_j, \tilde{\vec{\eta}}_i - \widetilde{\vec{\eta}}_j; \tilde{\vec{\xi}}_i, \tilde{\vec{\xi}}_j),$$

$$\widetilde{\vec{\kappa}}_i \cdot \vec{\partial}_{\tilde{\eta}_i} \widetilde{\mathcal{K}}_{ij} = \int d^3\sigma \Big[\Big(\widetilde{\vec{\kappa}}_i \cdot \vec{\partial}_{\tilde{\eta}_i} \vec{A}_{\perp Si}(\widetilde{\vec{\kappa}}_i, \widetilde{\vec{\kappa}}_j, \tilde{\vec{\eta}}_i - \widetilde{\vec{\eta}}_j; \tilde{\vec{\xi}}_i, \tilde{\vec{\xi}}_j) \Big) \cdot \vec{\pi}_{\perp Si}(\widetilde{\vec{\kappa}}_i, \widetilde{\vec{\kappa}}_j, \tilde{\vec{\eta}}_i - \widetilde{\vec{\eta}}_j; \tilde{\vec{\xi}}_i, \tilde{\vec{\xi}}_j) -$$
$$- \vec{A}_{\perp Si}(\widetilde{\vec{\kappa}}_i, \widetilde{\vec{\kappa}}_j, \tilde{\vec{\eta}}_i - \widetilde{\vec{\eta}}_j; \tilde{\vec{\xi}}_i, \tilde{\vec{\xi}}_j) \cdot \Big(\widetilde{\vec{\kappa}}_i \cdot \vec{\partial}_{\tilde{\eta}_i} \vec{\pi}_{\perp Si}(\widetilde{\vec{\kappa}}_i, \widetilde{\vec{\kappa}}_j, \tilde{\vec{\eta}}_i - \widetilde{\vec{\eta}}_j; \tilde{\vec{\xi}}_i, \tilde{\vec{\xi}}_j) \Big) \Big], \qquad (7.15)$$

But as we have seen in Eqs.(6.65) and (6.66), we can replace $\sum_{i<j} \vec{\partial}_{\eta_i} Q_i Q_j \mathcal{K}_{ij}$ by $-\int d^3\sigma [\vec{\pi}_{\perp S} \times \vec{B}_S](\tau, \vec{\sigma})$. Using the integral in Eq.(6.47), written in terms of canonical variables, we obtain that the old kinetic term can be rewritten as the new final kinetic term plus an extra scalar potential $U^{'}_{HDS}$ ($\eta_{ij} = |\vec{\eta}_{ij}| = |\vec{\eta}_i - \vec{\eta}_j|$)

$$\sum_{i=1}^N \sqrt{m_i^2 + \vec{\kappa}_i(\tau)^2} = \sqrt{m_i^2 + \widetilde{\vec{\kappa}}_i(\tau)^2} + U'_{HDS}(\tau),$$

$$U'_{HDS}(\tau) = -\sum_{i<j} \frac{Q_i Q_j}{8\pi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Big[\frac{\tilde{\vec{\mathbf{k}}}_i}{\sqrt{m_i^2 + \tilde{\vec{\kappa}}_i^2}} \cdot \frac{\tilde{\vec{\mathbf{k}}}_j}{\sqrt{m_j^2 + \tilde{\vec{\kappa}}_j^2}} \frac{1}{(2n+2m+2)!}$$
$$\Big((\frac{\tilde{\vec{\kappa}}_i}{\sqrt{m_i^2 + \tilde{\vec{\kappa}}_i^2}} \cdot \vec{\partial_{ij}})^{2m+2} (\frac{\tilde{\vec{\kappa}}_j}{\sqrt{m_j^2 + \tilde{\vec{\kappa}}_j^2}} \cdot \vec{\partial_{ij}})^{2n} +$$
$$+ 2(\frac{\tilde{\vec{\kappa}}_i}{\sqrt{m_i^2 + \tilde{\vec{\kappa}}_i^2}} \cdot \vec{\partial_{ij}})^{2m+1} (\frac{\tilde{\vec{\kappa}}_j}{\sqrt{m_j^2 + \vec{\kappa}_j^2}} \cdot \vec{\partial_{ij}})^{2n+1}$$
$$(\frac{\tilde{\vec{\kappa}}_i}{\sqrt{m_i^2 + \tilde{\vec{\kappa}}_i^2}} \cdot \vec{\partial_{ij}})^{2m} (\frac{\tilde{\vec{\kappa}}_j}{\sqrt{m_j^2 + \vec{\kappa}_j^2}} \cdot \vec{\partial_{ij}})^{2n+2} \Big) \tilde{\eta}_{ij}^{2n+2m+1} -$$

$$-\frac{1}{(2n+2m+4)!}\Big[(\frac{\tilde{\vec{\mathbf{k}}}_i}{\sqrt{m_i^2+\vec{\kappa}_i^2}}\cdot\vec{\partial}_{ij})(\frac{\tilde{\vec{\mathbf{l}}}_j}{\sqrt{m_j^2+\tilde{\vec{\kappa}}_j^2}}\cdot\vec{\partial}_{ij})$$

$$\Big((\frac{\tilde{\vec{\kappa}}_i}{\sqrt{m_i^2+\vec{\kappa}_i^2}}\cdot\vec{\partial}_{ij})^{2m+2}(\frac{\tilde{\vec{\kappa}}_j}{\sqrt{m_j^2+\tilde{\vec{\kappa}}_j^2}}\cdot\vec{\partial}_{ij})^{2n}+$$

$$+(\frac{\tilde{\vec{\kappa}}_i}{\sqrt{m_i^2+\tilde{\vec{\kappa}}_i^2}}\cdot\vec{\partial}_{ij})^{2m+1}(\frac{\tilde{\vec{\kappa}}_j}{\sqrt{m_j^2+\vec{\kappa}_j^2}}\cdot\vec{\partial}_{ij})^{2n+1}\Big)\tilde{\eta}_{ij}^{2n+2m+3}+$$

$$+(\frac{\tilde{\vec{\mathbf{l}}}_i}{\sqrt{m_i^2+\vec{\kappa}_i^2}}\cdot\vec{\partial}_{ij})(\frac{\tilde{\vec{\mathbf{k}}}_j}{\sqrt{m_j^2+\tilde{\vec{\kappa}}_j^2}}\cdot\vec{\partial}_{ij})\Big((\frac{\tilde{\vec{\kappa}}_i}{\sqrt{m_i^2+\tilde{\vec{\kappa}}_i^2}}\cdot\vec{\partial}_{ij})^{2m+1}(\frac{\tilde{\vec{\kappa}}_j}{\sqrt{m_j^2+\vec{\kappa}_j^2}}\cdot\vec{\partial}_{ij})^{2n+1}+$$

$$+(\frac{\tilde{\vec{\kappa}}_i}{\sqrt{m_i^2+\tilde{\vec{\kappa}}_i^2}}\cdot\vec{\partial}_{ij})^{2m}(\frac{\tilde{\vec{\kappa}}_j}{\sqrt{m_j^2+\vec{\kappa}_j^2}}\cdot\vec{\partial}_{ij})^{2n+2}\Big)\tilde{\eta}_{ij}^{2n+2m+3}\Big]-$$

$$-\frac{1}{(2n+2m+2)!}\Big(\frac{\tilde{\vec{\kappa}}_i}{\sqrt{m_i^2+\tilde{\vec{\kappa}}_i^2}}\cdot\frac{\tilde{\vec{\mathbf{k}}}_j}{\sqrt{m_j^2+\tilde{\vec{\kappa}}_j^2}}$$

$$[\frac{\tilde{\vec{\mathbf{k}}}_i-\tilde{\vec{\mathbf{l}}}_i}{\sqrt{m_i^2+\tilde{\vec{\kappa}}_i^2}}\cdot\vec{\partial}_{ij}(\frac{\tilde{\vec{\kappa}}_i}{\sqrt{m_i^2+\vec{\kappa}_i^2}}\cdot\vec{\partial}_{ij})^{2m+1}(\frac{\tilde{\vec{\kappa}}_j}{\sqrt{m_j^2+\tilde{\vec{\kappa}}_j^2}}\cdot\vec{\partial}_{ij})^{2n}+$$

$$+\frac{\tilde{\vec{\mathbf{k}}}_j-\tilde{\vec{\mathbf{l}}}_j}{\sqrt{m_j^2+\tilde{\vec{\kappa}}_j^2}}\cdot\vec{\partial}_{ij}(\frac{\tilde{\vec{\kappa}}_i}{\sqrt{m_i^2+\vec{\kappa}_i^2}}\cdot\vec{\partial}_{ij})^{2m}(\frac{\tilde{\vec{\kappa}}_j}{\sqrt{m_j^2+\tilde{\vec{\kappa}}_j^2}}\cdot\vec{\partial}_{ij})^{2n+1}]\Big)\tilde{\eta}_{ij}^{2n+2m+1}\Big]. \qquad (7.16)$$

There is some cancellations between the terms in $U^{'}_{HDS}(\tau)$ and those in $V_{HDS}(\tau)$ when expressed in the final variables.

The final result for the semi-classical Hamiltonian is

$$M=\mathcal{P}^{\tau}_{(int)}=\sum_{i=1}^N\sqrt{m_i^2+\widetilde{\vec{\kappa}_i}^2}+\sum_{i=1}^N\frac{\widetilde{\vec{\kappa}_i}\cdot\sum_{j\neq i}Q_iQ_j\vec{\partial}_{\tilde{\eta}_i}\mathcal{K}_{ij}}{2\sqrt{m_i^2+\widetilde{\vec{\kappa}_i}^2}}+\sum_{i\neq j}\frac{Q_iQ_j}{\widetilde{\vec{\eta}_i}-\widetilde{\vec{\eta}_j}\mid}+V_{DS}(\tau)=$$

$$=\sum_{i=1}^N\sqrt{m_i^2+\widetilde{\vec{\kappa}_i}^2}+\sum_{i\neq j}\frac{Q_iQ_j}{4\pi\mid\widetilde{\vec{\eta}_i}-\widetilde{\vec{\eta}_j}\mid}+\tilde{V}_{DS}(\tau),\qquad[\tilde{V}_{DS}\;=V_{DS}+U^{\prime}_{HDS}] \qquad (7.17)$$

where

$$\tilde{V}_{DS}=V_{LDS}+U_{HDS}+U^{\prime}_{HDS}:=V_{LDS}+V_{HDS}, \qquad (7.18)$$

with

$$V_{HDS}:=-\sum_{i<j}\frac{Q_iQ_j}{8\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Big[\frac{\tilde{\vec{\mathbf{k}}}_i}{\sqrt{m_i^2+\tilde{\vec{\kappa}}_i^2}}\cdot\frac{\tilde{\vec{\mathbf{k}}}_j}{\sqrt{m_j^2+\tilde{\vec{\kappa}}_j^2}}\frac{1}{(2n+2m+2)!}$$

$$\Big((\frac{\tilde{\vec{\kappa}}_i}{\sqrt{m_i^2+\tilde{\vec{\kappa}}_i^2}}\cdot\vec{\partial}_{ij})^{2m+2}(\frac{\tilde{\vec{\kappa}}_j}{\sqrt{m_j^2+\tilde{\vec{\kappa}}_j^2}}\cdot\vec{\partial}_{ij})^{2n}+$$

$$+\,(\frac{\tilde{\vec{\kappa}}_i}{\sqrt{m_i^2+\tilde{\vec{\kappa}}_i^2}}\cdot\vec{\partial_{ij}})^{2m}(\frac{\tilde{\vec{\kappa}}_j}{\sqrt{m_j^2+\tilde{\vec{\kappa}}_j^2}}\cdot\vec{\partial_{ij}})^{2n+2}\Big)\tilde{\eta}_{ij}^{2n+2m+1}-$$

$$-\frac{1}{(2n+2m+4)!}\Big[(\frac{\tilde{\vec{\mathbf{k}}}_i}{\sqrt{m_i^2+\tilde{\vec{\kappa}}_i^2}}\cdot\vec{\partial_{ij}})(\frac{\tilde{\vec{\mathbf{l}}}_j}{\sqrt{m_j^2+\tilde{\vec{\kappa}}_j^2}}\cdot\vec{\partial_{ij}})$$

$$\Big((\frac{\tilde{\vec{\kappa}}_i}{\sqrt{m_i^2+\tilde{\vec{\kappa}}_i^2}}\cdot\vec{\partial_{ij}})^{2m+2}(\frac{\tilde{\vec{\kappa}}_j}{\sqrt{m_j^2+\tilde{\vec{\kappa}}_j^2}}\cdot\vec{\partial_{ij}})^{2n}-$$

$$-(\frac{\tilde{\vec{\kappa}}_i}{\sqrt{m_i^2+\tilde{\vec{\kappa}}_i^2}}\cdot\vec{\partial_{ij}})^{2m+1}(\frac{\tilde{\vec{\kappa}}_j}{\sqrt{m_j^2+\tilde{\vec{\kappa}}_j^2}}\cdot\vec{\partial_{ij}})^{2n+1}\Big)\tilde{\eta}_{ij}^{2n+2m+3}+$$

$$+(\frac{\tilde{\vec{\mathbf{l}}}_i}{\sqrt{m_i^2+\tilde{\vec{\kappa}}_i^2}}\cdot\vec{\partial_{ij}})(\frac{\tilde{\vec{\mathbf{k}}}_j}{\sqrt{m_j^2+\tilde{\vec{\kappa}}_j^2}}\cdot\vec{\partial_{ij}})\Big((\frac{\tilde{\vec{\kappa}}_i}{\sqrt{m_i^2+\tilde{\vec{\kappa}}_i^2}}\cdot\vec{\partial_{ij}})^{2m}(\frac{\tilde{\vec{\kappa}}_j}{\sqrt{m_j^2+\tilde{\vec{\kappa}}_j^2}}\cdot\vec{\partial_{ij}})^{2n+2}-$$

$$-(\frac{\tilde{\vec{\kappa}}_i}{\sqrt{m_i^2+\tilde{\vec{\kappa}}_i^2}}\cdot\vec{\partial_{ij}})^{2m+1}(\frac{\tilde{\vec{\kappa}}_j}{\sqrt{m_j^2+\tilde{\vec{\kappa}}_j^2}}\cdot\vec{\partial_{ij}})^{2n+1}\Big)\tilde{\eta}_{ij}^{2n+2m+3}+$$

$$+\,2(\frac{\vec{\mathbf{l}}_{\mathbf{i}}}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\cdot\vec{\partial}_{ij})(\frac{\tilde{\vec{\mathbf{l}}}_j}{\sqrt{m_j^2+\tilde{\vec{\kappa}}_j^2}}\cdot\vec{\partial}_{ij})$$

$$(\frac{\tilde{\vec{\kappa}}_i}{\sqrt{m_i^2+\tilde{\vec{\kappa}}_i^2}}\cdot\vec{\partial}_{ij})^{2m+1}(\frac{\tilde{\vec{\kappa}}_j}{\sqrt{m_j^2+\tilde{\vec{\kappa}}_j^2}}\cdot\vec{\partial}_{ij})^{2n+1}\tilde{\eta}_{ij}^{2n+2m+3}\Big]-$$

$$-\frac{1}{(2n+2m+2)!}\Big(\frac{\tilde{\vec{\kappa}}_i}{\sqrt{m_i^2+\tilde{\vec{\kappa}}_i^2}}\cdot\frac{\tilde{\vec{\mathbf{k}}}_j}{\sqrt{m_j^2+\tilde{\vec{\kappa}}_j^2}}$$

$$[\frac{\tilde{\vec{\mathbf{k}}}_i-\tilde{\vec{\mathbf{l}}}_i}{\sqrt{m_i^2+\tilde{\vec{\kappa}}_i^2}}\cdot\vec{\partial_{ij}}(\frac{\tilde{\vec{\kappa}}_i}{\sqrt{m_i^2+\tilde{\vec{\kappa}}_i^2}}\cdot\vec{\partial_{ij}})^{2m+1}(\frac{\tilde{\vec{\kappa}}_j}{\sqrt{m_j^2+\tilde{\vec{\kappa}}_j^2}}\cdot\vec{\partial_{ij}})^{2n}]+$$

$$+\frac{\tilde{\vec{\mathbf{k}}}_i}{\sqrt{m_i^2+\tilde{\vec{\kappa}}_i^2}}\cdot\frac{\tilde{\vec{\kappa}}_j}{\sqrt{m_j^2+\tilde{\vec{\kappa}}_j^2}}$$

$$[\frac{\tilde{\vec{\mathbf{k}}}_j-\tilde{\vec{\mathbf{l}}}_j}{\sqrt{m_j^2+\tilde{\vec{\kappa}}_j^2}}\cdot\vec{\partial_{ij}}(\frac{\tilde{\vec{\kappa}}_i}{\sqrt{m_i^2+\tilde{\vec{\kappa}}_i^2}}\cdot\vec{\partial_{ij}})^{2m}(\frac{\tilde{\vec{\kappa}}_j}{\sqrt{m_j^2+\tilde{\vec{\kappa}}_j^2}}\cdot\vec{\partial_{ij}})^{2n+1}]\Big)\tilde{\eta}_{ij}^{2n+2m+1}-$$

$$-\frac{\vec{\mathbf{k}}_{\mathbf{i}}}{\sqrt{m_i^2+\tilde{\vec{\kappa}}_i^2}}\cdot\frac{\vec{\mathbf{k}}_{\mathbf{j}}}{\sqrt{m_j^2+\tilde{\vec{\kappa}}_j^2}}(\frac{\tilde{\vec{\kappa}}_i}{\sqrt{m_i^2+\tilde{\vec{\kappa}}_i^2}}\cdot\vec{\partial}_{ij})^{2m+2}(\frac{\tilde{\vec{\kappa}}_j}{\sqrt{m_j^2+\tilde{\vec{\kappa}}_j^2}}\cdot\vec{\partial}_{ij})^{2n+2}\frac{\eta_{ij}^{2n+2m+3}}{(2n+2m+4)!}+$$

$$
+(\frac{\vec{\mathbf{k}}_{\mathbf{i}}}{\sqrt{m_i^2+\tilde{\vec{\kappa}}_i^2}}\cdot\vec{\partial}_{ij})(\frac{\vec{\mathbf{k}}_{\mathbf{j}}}{\sqrt{m_j^2+\tilde{\vec{\kappa}}_j^2}}\cdot\vec{\partial}_{ij})
$$

$$
(\frac{\tilde{\vec{\kappa}}_i}{\sqrt{m_i^2+\tilde{\vec{\kappa}}_i^2}}\cdot\vec{\partial}_{ij})^{2m+2}(\frac{\tilde{\vec{\kappa}}_j}{\sqrt{m_j^2+\tilde{\vec{\kappa}}_j^2}}\cdot\vec{\partial}_{ij})^{2n+2}\frac{\eta_{ij}^{2n+2m+5}}{(2n+2m+6)!}\Big]. \tag{7.19}
$$

The term V_{LDS} is defined as in Eq.(7.12) but with all the old non-canonical variables replaced with the new canonical ones. Due to Grassmann truncation the expressions are equivalent. As we shall show in the next Section, it reduces in lowest order to the pseudo-classical version of the Breit results (of order $1/c^2$). The term V_{HDS} involves the double infinite series and is of order $1/c^4$ and higher. In general there is no closed form for it (see Section VI of Ref. [5] for related consideration for spinless charged particles), but in the case of the two body problem the double infinite series can be summed to closed form in the rest system.

VIII. THE SEMI-CLASSICAL 2-BODY PROBLEM: HYDROGEN ATOM, MUONIUM AND POSITRONIUM.

A. The Semi-Classical 2-Body Problem.

In this Section we specialize to the two body problem, which includes the pseudo-classical analogues of the hydrogen atom, muonium and positronium.

1) The rest frame form of the energy in the unequal mass case (*muonium*) is ($\tilde{\eta} := |\widetilde{\vec{\eta}_1} - \widetilde{\vec{\eta}_2}|$ and $\tilde{\vec{\kappa}} := \tilde{\vec{\kappa}}_1 = -\tilde{\vec{\kappa}}_2$)

$$M = \sqrt{m_1^2 + \tilde{\vec{\kappa}}^2} + \sqrt{m_2^2 + \tilde{\vec{\kappa}}^2} + \frac{Q_1 Q_2}{4\pi\tilde{\eta}} + \tilde{V}_{DS}(\tau), \tag{8.1}$$

where $\tilde{V}_{DS} = V_{LDS} + V_{HDS}$. From Eq.(7.12) we obtain for arbitrary masses

$$\begin{aligned}
V_{LDS} \; : \; &= \frac{Q_1 Q_2}{4\pi} \Big[\frac{\tilde{\vec{\kappa}}^2 + (\tilde{\vec{\kappa}} \cdot \hat{\eta})^2}{2\sqrt{m_1^2 + \tilde{\vec{\kappa}}^2}\sqrt{m_2^2 + \tilde{\vec{\kappa}}^2}} \frac{1}{\tilde{\eta}} - \\
&- i \frac{\tilde{\vec{\kappa}} \cdot \vec{\xi}_1 \vec{\xi}_1 \cdot \vec{\eta}}{\tilde{\eta}^3 (m_1 + \sqrt{m_1^2 + \tilde{\vec{\kappa}}^2}) \sqrt{m_1^2 + \tilde{\vec{\kappa}}^2}} - i \frac{\tilde{\vec{\kappa}} \cdot \vec{\xi}_2 \vec{\xi}_2 \cdot \vec{\eta}}{\tilde{\eta}^3 (m_2 + \sqrt{m_2^2 + \tilde{\vec{\kappa}}^2}) \sqrt{m_2^2 + \tilde{\vec{\kappa}}^2}} - \\
&- i \frac{\vec{\xi}_1 \times \vec{\xi}_1 \cdot (\tilde{\vec{\mathbf{k}}}_2 \times \vec{\partial}_\eta)}{2\sqrt{m_1^2 + \tilde{\vec{\kappa}}^2}} \frac{1}{\tilde{\eta}\sqrt{m_2^2 + (\tilde{\vec{\kappa}} \cdot \hat{\eta})^2}} + i \frac{\vec{\xi}_2 \times \vec{\xi}_2 \cdot (\tilde{\vec{\mathbf{k}}}_1 \times \vec{\partial}_\eta)}{2\sqrt{m_2^2 + \tilde{\vec{\kappa}}^2}} \frac{1}{\tilde{\eta}\sqrt{m_1^2 + (\tilde{\vec{\kappa}} \cdot \hat{\eta})^2}} + \\
&+ i \frac{\tilde{\vec{\kappa}} \cdot \vec{\xi}_1}{\sqrt{m_1^2 + \tilde{\vec{\kappa}}^2}(m_1 + \sqrt{m_1^2 + \tilde{\vec{\kappa}}^2})} \\
&\Big([\vec{\xi}_1 - \frac{i \tilde{\vec{\kappa}} \cdot \vec{\xi}_2 \vec{\xi}_1 \cdot \vec{\xi}_2 \vec{\partial}_\eta}{\sqrt{m_2^2 + \tilde{\vec{\kappa}}^2}(m_2 + \sqrt{m_2^2 + \tilde{\vec{\kappa}}^2})}] \cdot \frac{\hat{\eta}}{\tilde{\eta}^2} (\frac{m_2^2 \sqrt{m_2^2 + \tilde{\vec{\kappa}}^2}}{[m_2^2 + (\tilde{\vec{\kappa}} \cdot \hat{\eta})^2]^{3/2}} - 1) + \\
&+ i \vec{\xi}_1 \cdot \vec{\xi}_2 \vec{\xi}_2 \cdot \vec{\partial}_\eta \tilde{\vec{\kappa}} \cdot \vec{\partial}_\eta \frac{1}{\tilde{\eta}} \frac{1}{\sqrt{m_2^2 + (\tilde{\vec{\kappa}} \cdot \hat{\eta})^2}} \Big) + \\
&+ i \frac{\tilde{\vec{\kappa}} \cdot \vec{\xi}_2}{\sqrt{m_2^2 + \tilde{\vec{\kappa}}^2}(m_2 + \sqrt{m_2^2 + \tilde{\vec{\kappa}}^2})} \\
&\Big([\vec{\xi}_2 - \frac{i \tilde{\vec{\kappa}} \cdot \vec{\xi}_1 \vec{\xi}_2 \cdot \vec{\xi}_1 \vec{\partial}_\eta}{\sqrt{m_1^2 + \tilde{\vec{\kappa}}^2}(m_1 + \sqrt{m_1^2 + \tilde{\vec{\kappa}}^2})}] \cdot \frac{\hat{\eta}}{\tilde{\eta}^2} (\frac{m_1^2 \sqrt{m_1^2 + \tilde{\vec{\kappa}}^2}}{[m_1^2 + (\tilde{\vec{\kappa}} \cdot \hat{\eta})^2]^{3/2}} - 1) + \\
&+ i \vec{\xi}_2 \cdot \vec{\xi}_1 \vec{\xi}_1 \cdot \vec{\partial}_\eta \tilde{\vec{\kappa}} \cdot \vec{\partial}_\eta \frac{1}{\tilde{\eta}} \frac{1}{\sqrt{m_2^2 + (\tilde{\vec{\kappa}} \cdot \hat{\eta})^2}} \Big) + \\
&+ \frac{\vec{\mathbf{f}}_1 \cdot \tilde{\vec{\mathbf{k}}}_2}{\sqrt{m_2^2 + \tilde{\vec{\kappa}}^2}} \frac{1}{\tilde{\eta}\sqrt{m_1^2 + (\tilde{\vec{\kappa}} \cdot \hat{\eta})^2}} + \frac{\vec{\mathbf{f}}_2 \cdot \tilde{\vec{\mathbf{k}}}_1}{\sqrt{m_1^2 + \tilde{\vec{\kappa}}^2}} \frac{1}{\tilde{\eta}\sqrt{m_2^2 + (\tilde{\vec{\kappa}} \cdot \hat{\eta})^2}} -
\end{aligned}$$

$$
\begin{aligned}
&-\frac{\widetilde{\vec\kappa}\cdot\vec{\mathbf{f}}_2-\widetilde{\vec\kappa}\cdot\vec{\mathbf{f}}_1+\vec{\mathbf{f}}_1\cdot\vec{\mathbf{f}}_2}{2\sqrt{m_1^2+\widetilde{\vec\kappa}^2}\sqrt{m_2^2+\widetilde{\vec\kappa}^2}}\frac{1}{\tilde\eta}+\\
&+\Big(\widetilde{\vec\kappa}\cdot\vec{\mathbf{f}}_2-\widetilde{\vec\kappa}\cdot\vec{\mathbf{f}}_1+\vec{\mathbf{f}}_1\cdot\vec{\mathbf{f}}_2-(\vec{\mathbf{f}}_2\vec\kappa\cdot\hat\eta-\vec{\mathbf{f}}_1\widetilde{\vec\kappa}_j\cdot\hat\eta+\vec{\mathbf{f}}_2\vec{\mathbf{f}}_1\cdot\hat\eta)\cdot\hat\eta\Big)\\
&\quad\Big(\frac{1}{2\sqrt{m_1^2+\widetilde{\vec\kappa}^2}\sqrt{m_2^2+\widetilde{\vec\kappa}^2}\tilde\eta}-\frac{2}{\tilde\eta\sqrt{m_2^2+\widetilde{\vec\kappa}^2}(\sqrt{m_1^2+\widetilde{\vec\kappa}^2}+\sqrt{m_1^2+(\widetilde{\vec\kappa}\cdot\hat\eta)^2})}\Big)-\\
&\quad-2\Big(\widetilde{\vec\kappa}^2(\vec{\mathbf{f}}_2\cdot\widetilde{\vec\kappa}-(\vec{\mathbf{f}}_2\cdot\hat\eta)(\widetilde{\vec\kappa}_i\cdot\hat\eta))-\vec{\mathbf{f}}_2\cdot[(\widetilde{\vec\kappa}_i\cdot\hat\eta)(\widetilde{\vec\kappa}-\hat\eta(\widetilde{\vec\kappa}\cdot\hat\eta))](\widetilde{\vec\kappa}\cdot\hat\eta)-\\
&\quad-\widetilde{\vec\kappa}^2(\vec{\mathbf{f}}_1\cdot\widetilde{\vec\kappa}-(\vec{\mathbf{f}}_1\cdot\hat\eta)(\widetilde{\vec\kappa}\cdot\hat\eta))-\vec{\mathbf{f}}_1\cdot[(\widetilde{\vec\kappa}\cdot\hat\eta)(\widetilde{\vec\kappa}-\hat\eta(\widetilde{\vec\kappa}\cdot\hat\eta))](\widetilde{\vec\kappa}\cdot\hat\eta)-\\
&\quad-(\vec{\mathbf{f}}_1\cdot\widetilde{\vec\kappa})(\widetilde{\vec\kappa}\cdot\widetilde{\vec\kappa}-(\widetilde{\vec\kappa}\cdot\hat\eta)(\widetilde{\vec\kappa}\cdot\hat\eta))-\widetilde{\vec\kappa}\cdot[(\vec{\mathbf{f}}_1\cdot\hat\eta)(\widetilde{\vec\kappa}-\hat\eta(\widetilde{\vec\kappa}\cdot\hat\eta))](\widetilde{\vec\kappa}\cdot\hat\eta)+\\
&\quad+(\vec{\mathbf{f}}_2\cdot\widetilde{\vec\kappa})(\widetilde{\vec\kappa}\cdot\widetilde{\vec\kappa}-(\widetilde{\vec\kappa}\cdot\hat\eta)(\widetilde{\vec\kappa}\cdot\hat\eta))-\widetilde{\vec\kappa}\cdot[(\vec{\mathbf{f}}_2\cdot\hat\eta)(\widetilde{\vec\kappa}-\hat\eta(\widetilde{\vec\kappa}\cdot\hat\eta))](\widetilde{\vec\kappa}\cdot\hat\eta)+\\
&\quad+(\vec{\mathbf{f}}_1\cdot\widetilde{\vec\kappa})(\vec{\mathbf{f}}_2\cdot\vec\kappa-(\vec{\mathbf{f}}_2\cdot\hat\eta)(\widetilde{\vec\kappa}\cdot\hat\eta))-\vec{\mathbf{f}}_2\cdot[(\vec{\mathbf{f}}_1\cdot\hat\eta)(\widetilde{\vec\kappa}-\hat\eta(\widetilde{\vec\kappa}\cdot\hat\eta))](\widetilde{\vec\kappa}\cdot\hat\eta)\Big)\\
&\quad\frac{1}{\tilde\eta\sqrt{m_2^2+\widetilde{\vec\kappa}^2}\sqrt{m_1^2+(\widetilde{\vec\kappa}\cdot\hat\eta)^2}}\frac{1}{(\sqrt{m_1^2+\widetilde{\vec\kappa}^2}+\sqrt{m_1^2+(\widetilde{\vec\kappa}\cdot\hat\eta)^2})^2}+\\
&\quad+[\widetilde{\vec\kappa}\cdot\vec{\mathbf{g}}_1-\widetilde{\vec\kappa}\cdot(\vec{\mathbf{g}}_1\cdot(\hat\eta\hat\eta))]\frac{2}{\tilde\eta\sqrt{m_1^2+\widetilde{\vec\kappa}^2}\sqrt{m_2^2+(\widetilde{\vec\kappa}\cdot\hat\eta)^2}}\Big(\frac{\sqrt{m_2^2+\widetilde{\vec\kappa}^2}}{\sqrt{m_2^2+\widetilde{\vec\kappa}^2}+\sqrt{m_2^2+(\widetilde{\vec\kappa}\cdot\hat\eta)^2}}\Big)-\\
&\quad-[\widetilde{\vec\kappa}\cdot\vec{\mathbf{g}}_2-\widetilde{\vec\kappa}\cdot(\vec{\mathbf{g}}_2\cdot(\hat\eta\hat\eta))]\frac{2}{\tilde\eta\sqrt{m_2^2+\widetilde{\vec\kappa}^2}\sqrt{m_1^2+(\widetilde{\vec\kappa}\cdot\hat\eta)^2}}\Big(\frac{\sqrt{m_1^2+\widetilde{\vec\kappa}^2}}{\sqrt{m_1^2+\widetilde{\vec\kappa}^2}+\sqrt{m_1^2+(\widetilde{\vec\kappa}\cdot\hat\eta)^2}}\Big)\Big].
\end{aligned}
\tag{8.2}
$$

In Appendix E we obtain the result

$$
\begin{aligned}
&V_{HDS}=-\frac{Q_1Q_2}{8\pi(m_1^2-m_2^2)}\Big[\widetilde{\vec{\mathbf{k}}}_1\cdot\widetilde{\vec{\mathbf{k}}}_2\frac{1}{\tilde\eta}\\
&\Big(\sqrt{\frac{m_2^2++\widetilde{\vec\kappa}^2}{m_1^2+\widetilde{\vec\kappa}^2}}+\sqrt{\frac{m_1^2+\widetilde{\vec\kappa}^2}{m_2^2+\widetilde{\vec\kappa}^2}}\Big)\Big(\sqrt{\frac{m_2^2+\widetilde{\vec\kappa}^2}{m_2^2+(\widetilde{\vec\kappa}\cdot\hat\eta)^2}}-\sqrt{\frac{m_1^2+\widetilde{\vec\kappa}^2}{m_1^2+(\widetilde{\vec\kappa}\cdot\hat\eta)^2}}\Big)+\\
&+\Big[-\Big(\Big(\sqrt{\frac{m_2^2+\widetilde{\vec\kappa}^2}{m_1^2+\widetilde{\vec\kappa}^2}}-1\Big)\widetilde{\vec{\mathbf{k}}}_1\cdot\widetilde{\vec{\mathbf{l}}}_2+\Big(\sqrt{\frac{m_1^2+\widetilde{\vec\kappa}^2}{m_2^2+\widetilde{\vec\kappa}^2}}-1\Big)\widetilde{\vec{\mathbf{k}}}_2\cdot\widetilde{\vec{\mathbf{l}}}_1+\widetilde{\vec{\mathbf{l}}}_1\cdot\widetilde{\vec{\mathbf{l}}}_2\Big)\\
&\frac{1}{2}\Big(\frac{\sqrt{m_2^2+\widetilde{\vec\kappa}^2}-\sqrt{m_2^2+(\widetilde{\vec\kappa}\cdot\hat\eta)^2}}{\sqrt{m_2^2+\widetilde{\vec\kappa}^2}+\sqrt{m_2^2+(\widetilde{\vec\kappa}\cdot\hat\eta)^2}}-\frac{\sqrt{m_1^2+\widetilde{\vec\kappa}^2}-\sqrt{m_1^2+(\widetilde{\vec\kappa}\cdot\hat\eta)^2}}{\sqrt{m_1^2+\widetilde{\vec\kappa}^2}+\sqrt{m_1^2+(\widetilde{\vec\kappa}\cdot\hat\eta)^2}}\Big)+\\
&+\Big(\Big(\sqrt{\frac{m_2^2+\widetilde{\vec\kappa}^2}{m_1^2+\widetilde{\vec\kappa}^2}}-1\Big)\frac{\widetilde{\vec{\mathbf{k}}}_1\widetilde{\vec{\mathbf{l}}}_2\cdot\cdot\hat\eta\hat\eta\widetilde{\vec\kappa}^2}{2}+\Big(\sqrt{\frac{m_1^2+\widetilde{\vec\kappa}^2}{m_2^2+\widetilde{\vec\kappa}^2}}-1\Big)\frac{\widetilde{\vec{\mathbf{k}}}_2\cdot(\widetilde{\vec{\mathbf{l}}}_1\cdot\hat\eta\hat\eta)\widetilde{\vec\kappa}^2}{2}+\frac{\widetilde{\vec{\mathbf{l}}}_1\cdot(\widetilde{\vec{\mathbf{l}}}_2\cdot\hat\eta\hat\eta)\widetilde{\vec\kappa}^2}{2}\Big)\\
&\Big(\frac{1}{(\sqrt{m_2^2+\widetilde{\vec\kappa}^2}+\sqrt{m_2^2+(\widetilde{\vec\kappa}\cdot\hat\eta)^2})^2}-\frac{1}{(\sqrt{m_1^2+\widetilde{\vec\kappa}^2}+\sqrt{m_1^2+(\widetilde{\vec\kappa}\cdot\hat\eta)^2})^2}\Big)-
\end{aligned}
$$

$$-\Big((\sqrt{\frac{m_2^2+\tilde{\vec{\kappa}}^2}{m_1^2+\tilde{\vec{\kappa}}^2}}-1)[\tilde{\vec{\mathbf{k}}}_1\cdot\tilde{\vec{\kappa}}\tilde{\vec{\mathbf{l}}}_2\cdot\tilde{\vec{\kappa}}-\tilde{\vec{\mathbf{k}}}_1\cdot(\tilde{\vec{\mathbf{l}}}_2\cdot(\tilde{\vec{\kappa}}\hat{\eta}+\hat{\eta}\tilde{\vec{\kappa}}))(\tilde{\vec{\kappa}}\cdot\hat{\eta})]+$$

$$+(\sqrt{\frac{m_1^2+\tilde{\vec{\kappa}}^2}{m_2^2+\tilde{\vec{\kappa}}^2}}-1)[\tilde{\vec{\mathbf{k}}}_2\cdot\tilde{\vec{\kappa}}\tilde{\vec{\mathbf{l}}}_1\cdot\tilde{\vec{\kappa}}-\tilde{\vec{\mathbf{k}}}_2\cdot(\tilde{\vec{\mathbf{l}}}_1\cdot(\tilde{\vec{\kappa}}\hat{\eta}+\hat{\eta}\tilde{\vec{\kappa}}))(\tilde{\vec{\kappa}}\cdot\hat{\eta})]+$$

$$+[\tilde{\vec{\mathbf{l}}}_2\cdot\tilde{\vec{\kappa}}\tilde{\vec{\mathbf{l}}}_1\cdot\tilde{\vec{\kappa}}-\tilde{\vec{\mathbf{l}}}_2\cdot(\tilde{\vec{\mathbf{l}}}_1\cdot(\tilde{\vec{\kappa}}\hat{\eta}+\hat{\eta}\tilde{\vec{\kappa}}))(\tilde{\vec{\kappa}}\cdot\hat{\eta})]\Big)$$

$$\Big(\sqrt{\frac{m_2^2+\tilde{\vec{\kappa}}^2}{m_2^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}}\frac{1}{(\sqrt{m_2^2+\tilde{\vec{\kappa}}^2}+\sqrt{m_2^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2})^2}-$$

$$-\sqrt{\frac{m_1^2+\tilde{\vec{\kappa}}^2}{m_1^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}}\frac{1}{(\sqrt{m_1^2+\tilde{\vec{\kappa}}^2}+\sqrt{m_1^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2})^2}\Big)-$$

$$-\Big((\sqrt{\frac{m_2^2+\tilde{\vec{\kappa}}^2}{m_1^2+\tilde{\vec{\kappa}}^2}}-1)\tilde{\vec{\mathbf{k}}}_1\cdot(\tilde{\vec{\mathbf{l}}}_2\cdot\hat{\eta}\hat{\eta})(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2+$$

$$+(\sqrt{\frac{m_2^2+\tilde{\vec{\kappa}}^2}{m_1^2+\tilde{\vec{\kappa}}^2}}-1)\tilde{\vec{\mathbf{k}}}_1\cdot(\tilde{\vec{\mathbf{l}}}_2\cdot\hat{\eta}\hat{\eta})(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2+\tilde{\vec{\mathbf{l}}}_1\cdot(\tilde{\vec{\mathbf{l}}}_2\cdot\hat{\eta}\hat{\eta})(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2\Big)$$

$$\Big((\frac{1}{(\sqrt{m_2^2+\tilde{\vec{\kappa}}^2}+\sqrt{m_2^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2})^2})(2+\sqrt{\frac{m_2^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}{m_2^2+\tilde{\vec{\kappa}}^2}})-$$

$$-(\frac{1}{(\sqrt{m_1^2+\tilde{\vec{\kappa}}^2}+\sqrt{m_1^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2})^2})(2+\sqrt{\frac{m_1^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}{m_1^2+\tilde{\vec{\kappa}}^2}})\Big)\Big]\frac{1}{\tilde{\eta}}-$$

$$-\Big(\tilde{\vec{\kappa}}\cdot\tilde{\vec{\mathbf{k}}}_2\frac{(\tilde{\vec{\mathbf{k}}}_1-\tilde{\vec{\mathbf{l}}}_1)\cdot\tilde{\vec{\kappa}}-(\tilde{\vec{\mathbf{k}}}_1-\tilde{\vec{\mathbf{l}}}_1)\cdot\hat{\eta}\tilde{\vec{\kappa}}\cdot\hat{\eta}}{m_1^2+\tilde{\vec{\kappa}}^2}-\tilde{\vec{\kappa}}\cdot\tilde{\vec{\mathbf{k}}}_1\frac{(\tilde{\vec{\mathbf{k}}}_2-\tilde{\vec{\mathbf{l}}}_2)\cdot\tilde{\vec{\kappa}}-(\tilde{\vec{\mathbf{k}}}_2-\tilde{\vec{\mathbf{l}}}_2)\cdot\hat{\eta}\tilde{\vec{\kappa}}\cdot\hat{\eta}}{m_2^2+\tilde{\vec{\kappa}}^2}\Big)$$

$$\Big(\sqrt{\frac{1}{m_2^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}}\frac{\sqrt{m_1^2+\tilde{\vec{\kappa}}^2}\sqrt{m_2^2+\tilde{\vec{\kappa}}^2}}{\sqrt{m_2^2+\tilde{\vec{\kappa}}^2}+\sqrt{m_2^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}}-$$

$$-\sqrt{\frac{1}{m_1^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}}\frac{\sqrt{m_1^2+\tilde{\vec{\kappa}}^2}\sqrt{m_2^2+\tilde{\vec{\kappa}}^2}}{\sqrt{m_1^2+\tilde{\vec{\kappa}}^2}+\sqrt{m_1^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}}\Big)\frac{1}{\tilde{\eta}}$$

$$-\tilde{\vec{\mathbf{k}}}_1\cdot\tilde{\vec{\mathbf{k}}}_2\frac{1}{2\sqrt{m_1^2+\tilde{\vec{\kappa}}^2}\sqrt{m_2^2+\tilde{\vec{\kappa}}^2}\tilde{\eta}}\Big((\sqrt{m_2^2+\tilde{\vec{\kappa}}^2}-\sqrt{m_2^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2})^2(2+\sqrt{\frac{m_2^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}{m_2^2+\tilde{\vec{\kappa}}^2}})-$$

$$-(\sqrt{m_1^2+\tilde{\vec{\kappa}}^2}-\sqrt{m_1^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2})^2(2+\sqrt{\frac{m_1^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}{m_1^2+\tilde{\vec{\kappa}}^2}})\Big)-$$

$$
\begin{aligned}
&-\Big[-\tilde{\vec{\mathbf{k}}}_1\cdot\tilde{\vec{\mathbf{k}}}_2\Big(\frac{m_2^2+\tilde{\vec{\kappa}}^2}{2}\big(\frac{\sqrt{m_2^2+\tilde{\vec{\kappa}}^2}-\sqrt{m_2^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}}{\sqrt{m_2^2+\tilde{\vec{\kappa}}^2}+\sqrt{m_2^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}}\big)-\\
&-\frac{m_1^2+\tilde{\vec{\kappa}}^2}{2}\big(\frac{\sqrt{m_1^2+\tilde{\vec{\kappa}}^2}-\sqrt{m_1^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}}{\sqrt{m_1^2+\tilde{\vec{\kappa}}^2}+\sqrt{m_1^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}}\big)\Big)+\\
&+\Big(\tilde{\vec{\mathbf{k}}}_1\cdot\tilde{\vec{\kappa}}\tilde{\vec{\mathbf{k}}}_2\cdot\tilde{\vec{\kappa}}-\tilde{\vec{\mathbf{k}}}_1\cdot(\tilde{\vec{\mathbf{k}}}_2\cdot(\tilde{\vec{\kappa}}\hat{\eta}+\hat{\eta}\tilde{\vec{\kappa}}))(\tilde{\vec{\kappa}}\cdot\hat{\eta})-\tilde{\vec{\mathbf{k}}}_1\cdot(\tilde{\vec{\mathbf{k}}}_2\cdot\hat{\eta}\hat{\eta})\tilde{\vec{\kappa}}^2\Big)\\
&\Big(\frac{(\sqrt{m_2^2+\tilde{\vec{\kappa}}^2})^3}{\sqrt{m_2^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}}\frac{1}{(\sqrt{m_2^2+\tilde{\vec{\kappa}}^2}+\sqrt{m_2^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2})^2}-\\
&-\frac{(\sqrt{m_1^2+\tilde{\vec{\kappa}}^2})^3}{\sqrt{m_1^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}}\frac{1}{(\sqrt{m_1^2+\tilde{\vec{\kappa}}^2}+\sqrt{m_1^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2})^2}\Big)-\\
&-\tilde{\vec{\mathbf{k}}}_1\tilde{\vec{\mathbf{k}}}_2\cdot\cdot\hat{\eta}\hat{\eta}(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2\Big(\frac{\sqrt{m_2^2+\tilde{\vec{\kappa}}^2}(2\sqrt{m_2^2+\tilde{\vec{\kappa}}^2}+\sqrt{m_2^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}}{(\sqrt{m_2^2+\tilde{\vec{\kappa}}^2}+\sqrt{m_2^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2})^2}-\\
&-\frac{\sqrt{m_1^2+\tilde{\vec{\kappa}}^2}(2\sqrt{m_1^2+\tilde{\vec{\kappa}}^2}+\sqrt{m_1^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}}{(\sqrt{m_1^2+\tilde{\vec{\kappa}}^2}+\sqrt{m_1^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2})^2}\Big)\Big]\frac{1}{\sqrt{m_1^2+\tilde{\vec{\kappa}}^2}\sqrt{m_2^2+\tilde{\vec{\kappa}}^2}\tilde{\eta}}\Big]\Big]. \qquad (8.3)
\end{aligned}
$$

2) For the pseudo-classical version of *positronium* ($m_1=m_2$)

$$
M=2\sqrt{m^2+\tilde{\vec{\kappa}}^2}+\frac{Q_1Q_2}{4\pi\tilde{\eta}}+V_{LDS}+V_{HDS}, \qquad (8.4)
$$

where now

$$
\begin{aligned}
V_{LDS}\;:=\;&\frac{Q_1Q_2}{4\pi}\Big[\frac{\tilde{\vec{\kappa}}^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}{2(m^2+\tilde{\vec{\kappa}}^2)^2}\frac{1}{\tilde{\eta}}-i\frac{\tilde{\vec{\kappa}}\cdot\vec{\xi}_1\vec{\xi}_1\cdot\vec{\eta}+\tilde{\vec{\kappa}}\cdot\vec{\xi}_2\vec{\xi}_2\cdot\vec{\eta}}{\tilde{\eta}^3(m+\sqrt{m^2+\tilde{\vec{\kappa}}^2})\sqrt{m^2+\tilde{\vec{\kappa}}^2}}-\\
&-i\frac{\vec{\xi}_1\times\vec{\xi}_1\cdot(\tilde{\vec{\mathbf{k}}}_2\times\vec{\partial}_\eta)-\vec{\xi}_2\times\vec{\xi}_2\cdot(\tilde{\vec{\mathbf{k}}}_1\times\vec{\partial}_\eta)}{2\sqrt{m^2+\tilde{\vec{\kappa}}^2}}\frac{1}{\tilde{\eta}\sqrt{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}}+\\
&+i\frac{\tilde{\vec{\kappa}}\cdot\vec{\xi}_1}{\sqrt{m^2+\tilde{\vec{\kappa}}^2}(m+\sqrt{m^2+\tilde{\vec{\kappa}}^2})}\\
&\cdot\Big([\vec{\xi}_1-\frac{i\tilde{\vec{\kappa}}\cdot\vec{\xi}_2\vec{\xi}_1\cdot\vec{\xi}_2\vec{\partial}_\eta}{\sqrt{m^2+\tilde{\vec{\kappa}}^2}(m+\sqrt{m^2+\tilde{\vec{\kappa}}^2})}]\cdot\frac{\hat{\eta}}{\tilde{\eta}^2}\big(\frac{m^2\sqrt{m^2+\tilde{\vec{\kappa}}^2}}{[m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2]^{3/2}}-1\big)+\\
&+i\vec{\xi}_1\cdot\vec{\xi}_2\vec{\xi}_2\cdot\vec{\partial}_\eta\tilde{\vec{\kappa}}\cdot\vec{\partial}_\eta\frac{1}{\tilde{\eta}}\frac{1}{\sqrt{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}}\Big)+\\
&+i\frac{\tilde{\vec{\kappa}}\cdot\vec{\xi}_2}{\sqrt{m^2+\tilde{\vec{\kappa}}^2}(m+\sqrt{m^2+\tilde{\vec{\kappa}}^2})}\\
&\cdot\Big([\vec{\xi}_2-\frac{i\tilde{\vec{\kappa}}\cdot\vec{\xi}_1\vec{\xi}_2\cdot\vec{\xi}_1\vec{\partial}_\eta}{\sqrt{m^2+\tilde{\vec{\kappa}}^2}(m+\sqrt{m^2+\tilde{\vec{\kappa}}^2})}]\cdot\frac{\hat{\eta}}{\tilde{\eta}^2}\big(\frac{m^2\sqrt{m^2+\tilde{\vec{\kappa}}^2}}{[m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2]^{3/2}}-1\big)+
\end{aligned}
$$

$$
\begin{aligned}
&+i\vec{\xi}_2\cdot\vec{\xi}_1\vec{\xi}_1\cdot\vec{\partial}_\eta\tilde{\vec{\kappa}}\cdot\vec{\partial}_\eta\frac{1}{\tilde{\eta}}\frac{1}{\sqrt{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}}\Big)+\\
&+\frac{\vec{\mathbf{f}}_{\mathbf{1}}\cdot\tilde{\vec{\mathbf{k}}}_2}{\sqrt{m^2+\tilde{\vec{\kappa}}^2}}\frac{1}{\tilde{\eta}\sqrt{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}}+\frac{\vec{\mathbf{f}}_{\mathbf{2}}\cdot\tilde{\vec{\mathbf{k}}}_1}{\sqrt{m^2+\tilde{\vec{\kappa}}^2}}\frac{1}{\tilde{\eta}\sqrt{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}}-\\
&-\frac{\tilde{\vec{\kappa}}\cdot\vec{\mathbf{f}}_2-\tilde{\vec{\kappa}}\cdot\vec{\mathbf{f}}_1+\vec{\mathbf{f}}_1\cdot\vec{\mathbf{f}}_2}{2\sqrt{m^2+\tilde{\vec{\kappa}}^2}\sqrt{m^2+\tilde{\vec{\kappa}}^2}}\frac{1}{\tilde{\eta}}+\\
&+[\tilde{\vec{\kappa}}\cdot\vec{\mathbf{f}}_2-\tilde{\vec{\kappa}}\cdot\vec{\mathbf{f}}_1+\vec{\mathbf{f}}_1\cdot\vec{\mathbf{f}}_2-(\vec{\mathbf{f}}_2\vec{\kappa}\cdot\hat{\eta}-\vec{\mathbf{f}}_1\tilde{\vec{\kappa}}_j\cdot\hat{\eta}+\vec{\mathbf{f}}_2\vec{\mathbf{f}}_1\cdot\hat{\eta})\cdot\hat{\eta}]\\
&\Big(\frac{1}{2(m^2+\tilde{\vec{\kappa}}^2)\tilde{\eta}}-\frac{2}{\tilde{\eta}\sqrt{m^2+\tilde{\vec{\kappa}}_j{}^2}\Big(\sqrt{m^2+\tilde{\vec{\kappa}}^2}+\sqrt{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}\Big)}\Big)-\\
&-2\Big(\tilde{\vec{\kappa}}^2(\vec{\mathbf{f}}_2\cdot\tilde{\vec{\kappa}}-(\vec{\mathbf{f}}_2\cdot\hat{\eta})(\tilde{\vec{\kappa}}_i\cdot\hat{\eta}))-\vec{\mathbf{f}}_2\cdot[(\tilde{\vec{\kappa}}_i\cdot\hat{\eta})(\tilde{\vec{\kappa}}-\hat{\eta}(\tilde{\vec{\kappa}}\cdot\hat{\eta}))](\tilde{\vec{\kappa}}\cdot\hat{\eta})-\\
&-\tilde{\vec{\kappa}}^2(\vec{\mathbf{f}}_1\cdot\tilde{\vec{\kappa}}-(\vec{\mathbf{f}}_1\cdot\hat{\eta})(\tilde{\vec{\kappa}}\cdot\hat{\eta}))-\vec{\mathbf{f}}_1\cdot[(\tilde{\vec{\kappa}}\cdot\hat{\eta})(\tilde{\vec{\kappa}}-\hat{\eta}(\tilde{\vec{\kappa}}\cdot\hat{\eta}))](\tilde{\vec{\kappa}}\cdot\hat{\eta})-\\
&-(\vec{\mathbf{f}}_1\cdot\tilde{\vec{\kappa}})(\tilde{\vec{\kappa}}\cdot\tilde{\vec{\kappa}}-(\tilde{\vec{\kappa}}\cdot\hat{\eta})(\tilde{\vec{\kappa}}\cdot\hat{\eta}))-\tilde{\vec{\kappa}}\cdot[(\vec{\mathbf{f}}_1\cdot\hat{\eta})(\tilde{\vec{\kappa}}-\hat{\eta}(\tilde{\vec{\kappa}}\cdot\hat{\eta}))](\tilde{\vec{\kappa}}\cdot\hat{\eta})+\\
&+(\vec{\mathbf{f}}_2\cdot\tilde{\vec{\kappa}})(\tilde{\vec{\kappa}}\cdot\tilde{\vec{\kappa}}-(\tilde{\vec{\kappa}}\cdot\hat{\eta})(\tilde{\vec{\kappa}}\cdot\hat{\eta}))-\tilde{\vec{\kappa}}\cdot[(\vec{\mathbf{f}}_2\cdot\hat{\eta})(\tilde{\vec{\kappa}}-\hat{\eta}(\tilde{\vec{\kappa}}\cdot\hat{\eta}))](\tilde{\vec{\kappa}}\cdot\hat{\eta})+\\
&+(\vec{\mathbf{f}}_1\cdot\tilde{\vec{\kappa}})(\vec{\mathbf{f}}_2\cdot\vec{\kappa}-(\vec{\mathbf{f}}_2\cdot\hat{\eta})(\tilde{\vec{\kappa}}\cdot\hat{\eta}))-\vec{\mathbf{f}}_2\cdot[(\vec{\mathbf{f}}_1\cdot\hat{\eta})(\tilde{\vec{\kappa}}-\hat{\eta}(\tilde{\vec{\kappa}}\cdot\hat{\eta}))](\tilde{\vec{\kappa}}\cdot\hat{\eta})\Big)\\
&\frac{1}{\tilde{\eta}\sqrt{m^2+\tilde{\vec{\kappa}}^2}\sqrt{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}}\frac{1}{(\sqrt{m^2+\tilde{\vec{\kappa}}^2}+\sqrt{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2})^2}+\\
&+\Big([\tilde{\vec{\kappa}}\cdot\vec{\mathbf{g}}_1-\tilde{\vec{\kappa}}\cdot(\vec{\mathbf{g}}_1\cdot(\hat{\eta}\hat{\eta}))]-[\tilde{\vec{\kappa}}\cdot\vec{\mathbf{g}}_2-\tilde{\vec{\kappa}}\cdot(\vec{\mathbf{g}}_2\cdot(\hat{\eta}\hat{\eta}))]\Big)\\
&\frac{2}{\tilde{\eta}\sqrt{m^2+\tilde{\vec{\kappa}}^2}\sqrt{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}}\Big(\frac{\sqrt{m^2+\tilde{\vec{\kappa}}^2}}{\sqrt{m^2+\tilde{\vec{\kappa}}^2}+\sqrt{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}}\Big)\Big].
\end{aligned}
\tag{8.5}
$$

while

$$
\begin{aligned}
V_{HDS}=-\frac{Q_1Q_2}{8\pi}\Big[&\tilde{\vec{\mathbf{k}}}_1\cdot\tilde{\vec{\mathbf{k}}}_2\frac{1}{\tilde{\eta}}\Big(\frac{1}{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}-\frac{1}{m^2+\tilde{\vec{\kappa}}^2}\Big)+\\
&+\Big(-\tilde{\vec{\mathbf{l}}}_1\cdot\tilde{\vec{\mathbf{l}}}_2\frac{1}{2}\frac{\sqrt{m^2+\tilde{\vec{\kappa}}^2}-\sqrt{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}}{\sqrt{m^2+\tilde{\vec{\kappa}}^2}+\sqrt{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}}\frac{1}{\sqrt{m^2+\tilde{\vec{\kappa}}^2}\sqrt{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}}+\\
&+\frac{\tilde{\vec{\mathbf{l}}}_1\cdot(\tilde{\vec{\mathbf{l}}}_2\cdot\hat{\eta}\hat{\eta})\tilde{\vec{\kappa}}^2}{2}\Big(\frac{1}{(\sqrt{m^2+\tilde{\vec{\kappa}}^2}+\sqrt{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2})^2}\Big)\frac{1}{\sqrt{m^2+\tilde{\vec{\kappa}}^2}\sqrt{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}}-\\
&-\Big(\tilde{\vec{\mathbf{l}}}_2\cdot\tilde{\vec{\kappa}}\tilde{\vec{\mathbf{l}}}_1\cdot\tilde{\vec{\kappa}}-\tilde{\vec{\mathbf{l}}}_2\cdot(\tilde{\vec{\mathbf{l}}}_1\cdot(\tilde{\vec{\kappa}}\hat{\eta}+\hat{\eta}\tilde{\vec{\kappa}})(\tilde{\vec{\kappa}}\cdot\hat{\eta}))\Big)\\
&\Big[\sqrt{\frac{m^2+\tilde{\vec{\kappa}}^2}{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}}\frac{1}{(\sqrt{m^2+\tilde{\vec{\kappa}}^2}+\sqrt{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2})^2}\\
&\Big(\frac{1}{\sqrt{m^2+\tilde{\vec{\kappa}}^2}\sqrt{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}}-\frac{1}{2}\Big(\frac{1}{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}-\frac{1}{m^2+\tilde{\vec{\kappa}}^2}\Big)\Big]-
\end{aligned}
$$

$$
-\tilde{\vec{\mathbf{l}}}_1\tilde{\vec{\mathbf{l}}}_2\cdot\cdot\hat{\eta}\hat{\eta}(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2(\frac{1}{(\sqrt{m^2+\tilde{\vec{\kappa}}^2}+\sqrt{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2})^2})(\frac{2}{\sqrt{m^2+\tilde{\vec{\kappa}}^2}\sqrt{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}}-
$$

$$
-\frac{1}{2}\sqrt{\frac{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}{m^2+\tilde{\vec{\kappa}}^2}}(\frac{1}{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}-\frac{1}{m^2+\tilde{\vec{\kappa}}^2})))\Big)\frac{1}{\tilde{\eta}}+
$$

$$
+\Big(\tilde{\vec{\kappa}}\cdot\tilde{\vec{\mathbf{k}}}_2(\tilde{\vec{\mathbf{l}}}_1\cdot\tilde{\vec{\kappa}}-(\tilde{\vec{\mathbf{k}}}_1-\tilde{\vec{\mathbf{l}}}_1)\cdot\hat{\eta}\tilde{\vec{\kappa}}\cdot\hat{\eta})-\tilde{\vec{\kappa}}\cdot\tilde{\vec{\mathbf{k}}}_1(\tilde{\vec{\mathbf{l}}}_2\cdot\tilde{\vec{\kappa}}-(\tilde{\vec{\mathbf{k}}}_2-\tilde{\vec{\mathbf{l}}}_2)\cdot\hat{\eta}\tilde{\vec{\kappa}}\cdot\hat{\eta})\Big]
$$

$$
\Big(\sqrt{\frac{1}{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}}\frac{1}{\sqrt{m^2+\tilde{\vec{\kappa}}^2}+\sqrt{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}}
$$

$$
(\frac{1}{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}+\frac{1}{\sqrt{m^2+\tilde{\vec{\kappa}}^2}\sqrt{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}}))\Big)\frac{1}{2\tilde{\eta}}+
$$

$$
+\frac{\tilde{\vec{\mathbf{k}}}_1\cdot\tilde{\vec{\mathbf{k}}}_2}{4(m^2+\tilde{\vec{\kappa}}^2)}\Big((\sqrt{m^2+\tilde{\vec{\kappa}}^2}-\sqrt{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2})^2\sqrt{\frac{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}{m^2+\tilde{\vec{\kappa}}^2}})
$$

$$
(\frac{1}{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}-\frac{1}{m^2+\tilde{\vec{\kappa}}^2})\frac{1}{\tilde{\eta}}-
$$

$$
-4(\sqrt{m^2+\tilde{\vec{\kappa}}^2}-\sqrt{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2})(\frac{1}{\sqrt{m^2+\tilde{\vec{\kappa}}^2}}-\frac{1}{\sqrt{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}}))\Big)-
$$

$$
-\Big(\tilde{\vec{\mathbf{k}}}_1\cdot\tilde{\vec{\mathbf{k}}}_2\Big(\frac{1}{2(\sqrt{m_2^2+\tilde{\vec{\kappa}}^2}+\sqrt{m_2^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2})}
$$

$$
(2\sqrt{m^2+\tilde{\vec{\kappa}}^2}-\sqrt{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}-\frac{m^2+\tilde{\vec{\kappa}}^2}{\sqrt{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}}))\Big)-
$$

$$
-\Big(\tilde{\vec{\mathbf{k}}}_1\cdot\tilde{\vec{\kappa}}\tilde{\vec{\mathbf{k}}}_2\cdot\tilde{\vec{\kappa}}-\tilde{\vec{\mathbf{k}}}_1\cdot(\tilde{\vec{\mathbf{k}}}_2\cdot(\tilde{\vec{\kappa}}\hat{\eta}+\hat{\eta}\tilde{\vec{\kappa}}))(\tilde{\vec{\kappa}}\cdot\hat{\eta})-\tilde{\vec{\mathbf{k}}}_1\tilde{\vec{\mathbf{k}}}_2\cdot\cdot\hat{\eta}\hat{\eta}\tilde{\vec{\kappa}}^2\Big)
$$

$$
(\frac{1}{(\sqrt{m^2+\tilde{\vec{\kappa}}^2}+\sqrt{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2})^2})
$$

$$
\Big(3\sqrt{m^2+\tilde{\vec{\kappa}}^2}-\frac{m^2+\tilde{\vec{\kappa}}^2}{\sqrt{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}}-\frac{(\sqrt{m^2+\tilde{\vec{\kappa}}^2})^3}{2(m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2)}\Big)+
$$

$$
+\tilde{\vec{\mathbf{k}}}_1\cdot(\tilde{\vec{\mathbf{k}}}_2\cdot\hat{\eta}\hat{\eta})(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2[\frac{\sqrt{m^2+\tilde{\vec{\kappa}}^2}}{(\sqrt{m^2+\tilde{\vec{\kappa}}^2}+\sqrt{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2})^2}
$$

$$
(\frac{2}{\sqrt{m^2+\tilde{\vec{\kappa}}^2}}-\frac{1}{\sqrt{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}}+\frac{\sqrt{m^2+(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2}}{2(m^2+\tilde{\vec{\kappa}}^2)})]\Big)\frac{1}{(m^2+\tilde{\vec{\kappa}}^2)\tilde{\eta}}\Big]. \qquad (8.6)
$$

3) For the pseudo-classical version of *hydrogen-like atoms* ($m_2\rightarrow\infty$) the active part of the rest energy is just

$$
M=\sqrt{m_1^2+\tilde{\vec{\kappa}}^2}+\frac{Q_1Q_2}{4\pi\tilde{\eta}}+V_{LDS}, \qquad (8.7)
$$

since $V_{HDS}\rightarrow 0$ in this limit. In this case V_{LDS} simplifies dramatically to just

$$V_{LDS}=\frac{Q_1Q_2}{4\pi}[-i\frac{\tilde{\vec{\kappa}}\cdot\vec{\xi}_1\vec{\xi}_1\cdot\vec{\eta}}{\tilde{\eta}^3(m_1+\sqrt{m_1^2+\tilde{\vec{\kappa}}^2})\sqrt{m_1^2+\tilde{\vec{\kappa}}^2}}]. \tag{8.8}$$

In the remaining part of this section we check the lowest order portion for comparison with the known *Breit results.* At lowest order ($O(1/c^2)$) Eq.(7.12)

$$\begin{aligned}V_{LDS}(\tau)=\sum_{i\neq j}\frac{Q_iQ_j}{4\pi}\Big[&-\frac{\vec{\kappa}_i\cdot\vec{\kappa}_j+\vec{\kappa}_i\cdot\hat{\eta}_{ij}\vec{\kappa}_j\cdot\hat{\eta}_{ij}}{4m_im_j}\frac{1}{\eta_{ij}}-i\frac{\vec{\kappa}_i\cdot\vec{\xi}_i\vec{\xi}_1\cdot\vec{\eta}_{ij}}{2\eta_{ij}^3m_i^2}-\\&-i\vec{\xi}_i\times\vec{\xi}_i\cdot(\vec{\kappa}_j\times\vec{\partial}_{ij})\frac{1}{2\eta_{ij}m_im_j}+\frac{\vec{\mathbf{f}}_\mathbf{i}\cdot\vec{\mathbf{f}}_\mathbf{j}}{2m_im_j}\frac{1}{\eta_{ij}}+\\&+\Big(\vec{\kappa}_i\cdot\vec{\mathbf{f}}_j+\vec{\kappa}_j\cdot\hat{\vec{f}}_i+\vec{\mathbf{f}}_i\cdot\vec{\mathbf{f}}_j-(\vec{\mathbf{f}}_j\vec{\kappa}_i\cdot\hat{\eta}_{ij}+\vec{\mathbf{f}}_i\vec{\kappa}_j\cdot\hat{\eta}_{ij}+\vec{\mathbf{f}}_j\vec{\mathbf{f}}_i\cdot\hat{\eta}_{ij})\cdot\hat{\eta}_{ij}\Big)[\frac{-1}{2m_im_j\eta_{ij}}]+\\&+\Big(\vec{\kappa}_i\cdot\vec{\mathbf{g}}_j-\vec{\kappa}_i\cdot(\hat{\vec{g}}_j\cdot(\hat{\eta}_{ij}\hat{\eta}_{ij}))\Big)\frac{1}{\eta_{ij}m_im_j}\Big].\end{aligned} \tag{8.9}$$

To lowest order $\vec{\mathbf{f}}_i=-i\vec{\xi}_i\vec{\xi}_i\cdot\vec{\partial}_{ij}=i\vec{\xi}_i\cdot\vec{\partial}_{ij}\vec{\xi}_i,\ \hat{\vec{g}}_i=0$ so

$$\begin{aligned}V_{LDS}(\tau)=\sum_{i\neq j}\frac{Q_iQ_j}{4\pi}\Big[&-\frac{\vec{\kappa}_i\cdot\vec{\kappa}_j+\vec{\kappa}_i\cdot\hat{\eta}_{ij}\vec{\kappa}_j\cdot\hat{\eta}_{ij}}{4m_im_j}\frac{1}{\eta_{ij}}-i\frac{\vec{\kappa}_i\cdot\vec{\xi}_i\vec{\xi}_i\cdot\vec{\eta}_{ij}}{2\eta_{ij}^3m_i^2}-\\&-i\vec{\xi}_i\times\vec{\xi}_i\cdot(\vec{\kappa}_j\times\vec{\partial}_{ij})\frac{1}{2\eta_{ij}m_im_j}+\vec{\xi}_i\times\vec{\xi}_i\cdot(\vec{\xi}_2\cdot\vec{\partial}_{ij}\vec{\xi}_j\times\vec{\partial}_{ij})\frac{1}{2\eta_{ij}m_im_j}+\\&+\frac{(\vec{\xi}_i\cdot\vec{\partial}_{ij})(\vec{\xi}_i\cdot\vec{\xi}_j)(\vec{\xi}_2\cdot\vec{\partial}_{ij})}{2m_im_j}\frac{1}{\eta_{ij}}+\\&+[-2i\vec{\kappa}_i\cdot\vec{\xi}_2\vec{\xi}_2\cdot\vec{\partial}_{ij}+(\vec{\xi}_i\cdot\vec{\partial}_{ij})(\vec{\xi}_i\cdot\vec{\xi}_j)\vec{\xi}_2\cdot\vec{\partial}_{ij}-\\&-(2i\vec{\xi}_2\cdot\vec{\partial}_{ij}\vec{\xi}_2\cdot\hat{\eta}_{ij}\vec{\kappa}_i\cdot\hat{\eta}_{ij}+\vec{\xi}_2\cdot\vec{\partial}_{ij}\vec{\xi}_2\cdot\hat{\eta}_{ij}\vec{\xi}_i\cdot\vec{\partial}_{ij}\vec{\xi}_i\cdot\hat{\eta}_{ij})][\frac{-1}{2m_im_j\eta_{ij}}]\Big].\end{aligned} \tag{8.10}$$

Including the derivative terms and canceling like terms we obtain our lowest order pseudo-classical expression, giving the semi-relativistic (order $1/c^2$) potential energy terms for the pseudo-classical analogue of *hydrogen-like systems* (either mass $\rightarrow\infty$), *positronium-like systems* ($m_1=m_2$), or general unequal mass systems (*muonium -like*):

$$\begin{aligned}V_{LDS}(\tau)=\sum_{i\neq j}\frac{Q_iQ_j}{4\pi}\Big[&-\frac{\vec{\kappa}_i\cdot\vec{\kappa}_j+\vec{\kappa}_i\cdot\hat{\eta}_{ij}\vec{\kappa}_j\cdot\hat{\eta}_{ij}}{4m_im_j}\frac{1}{\eta_{ij}}-i\frac{\vec{\kappa}_i\cdot\vec{\xi}_i\vec{\xi}_i\cdot\vec{\eta}_{ij}}{2\eta_{ij}^3m_i^2}-\\&-i\frac{\vec{\xi}_i\times\vec{\xi}_i\cdot(\vec{\eta}_{ij}\times\vec{\kappa}_j)}{2\eta_{ij}^3m_im_j}+\frac{\vec{\xi}_i\times\vec{\xi}_i\cdot\vec{\xi}_2\times\vec{\xi}_2}{2m_im_j}(\frac{1}{\eta_{ij}^3}+\frac{4\pi\delta(\vec{\eta}_{ij})}{3})-\\&-\frac{3\vec{\xi}_i\times\vec{\xi}_i\cdot\vec{\xi}_2\times\hat{\eta}_{ij}(\vec{\xi}_j\cdot\hat{\eta}_{ij})}{2\eta_{ij}^3m_im_j}\Big].\end{aligned} \tag{8.11}$$

B. Quantization of the Lowest Order Potential.

Next we examine the quantum version of this interaction for comparison with the standard results of the *reduction of the Bethe-Salpeter equation.* In the manifestly Lorentz covariant formulation with the two signs of the energy the 5 Grassmann variables $\xi^\mu$and ξ_5 are quantized to $\gamma_5\gamma^\mu/\sqrt{2}$and $\gamma_5/\sqrt{2}$,so that the Dirac constraint $p_\mu\xi^\mu - m\xi_5 \approx 0$ becomes the wave operator $\gamma_5[p_\mu\gamma^\mu - m]$. But now with the positive-energy spinning particles we have only 3 Grassmann variables ξ^r and no spinor wave equation (the spinors satisfy only the Klein-Gordon equation). Therefore we have a different quantization (corresponding to the pseudo-classical spin $\vec{S} = (-i/2)\vec{\xi} \times \vec{\xi}$ going into $\vec{\sigma}/2$) holding only on the positive-energy branch of the mass spectrum. The quantization is of Eq.(6.64), $\{\xi^h_i, \xi^k_i\} = -i\delta^{hk}$, is

$$\xi^r \mapsto \frac{\sigma_r}{\sqrt{2}}, \tag{8.12}$$

and is compatible with the Pauli matrix algebra of

$$\sigma^h\sigma^k + \sigma^k\sigma^h = 2\delta^{hk}. \tag{8.13}$$

With this conversion rules and with the algebra of the Pauli matrices we obtain

$$\begin{aligned}
\vec{A}\cdot\vec{\xi}_i\vec{B}\cdot\vec{\xi}_i &= \frac{1}{2}\vec{A}\cdot\vec{B} + \frac{i}{2}\vec{A}\times\vec{B}\cdot\vec{\sigma},\\
\vec{\xi}_i\times\vec{\xi}_i &= +i\vec{\sigma}_i,\\
(\vec{A}\cdot\vec{\xi}_i)(\vec{B}\times\vec{\xi}_i) &= -\frac{1}{2}(\vec{A}\times\vec{B}) - \frac{i}{2}\vec{A}(\vec{B}\cdot\vec{\sigma}_i) + \frac{i}{2}(\vec{A}\cdot\vec{B})\vec{\sigma}_i,
\end{aligned} \tag{8.14}$$

giving

$$\begin{aligned}
V_{LDS}(\tau) = \sum_{i\neq j}\frac{Q_iQ_j}{4\pi}\Big[\Big(&-\frac{\vec{\kappa}_i\cdot\vec{\kappa}_j + \vec{\kappa}_i\cdot\hat{\eta}_{ij}\vec{\kappa}_j\cdot\hat{\eta}_{ij}}{4m_im_j\eta_{ij}} - i\frac{\vec{\kappa}_i\cdot\vec{\eta}_{ij}}{4\eta^3_{ij}m^2_i}\Big)_{ordered} -\\
&-\frac{\vec{\eta}_{ij}\times\vec{\kappa}_i\cdot\vec{\sigma}_i}{4\eta^3_{ij}m^2_i} + \frac{\vec{\eta}_{ij}\times\vec{\kappa}_j\cdot\vec{\sigma}_i}{2\eta^3_{ij}m_im_j} -\\
&-\frac{\vec{\sigma}_i\cdot\vec{\sigma}_j}{2m_im_j}(\frac{1}{\eta^3_{ij}} + \frac{4\pi\delta(\vec{\eta}_{ij})}{3}) + \frac{3(\vec{\sigma}_i\cdot\vec{\sigma}_j - \vec{\sigma}_i\cdot\hat{\eta}_{ij}\vec{\sigma}_j\cdot\hat{\eta}_{ij})}{4m_im_j\eta^3_{ij}}\Big].
\end{aligned} \tag{8.15}$$

For two particles in the CM system the above reduces to

$$\begin{aligned}
V_{LDS}(\tau) = \frac{Q_1Q_2}{4\pi}\Big[\Big(&\frac{\vec{\kappa}^2 + (\vec{\kappa}\cdot\hat{\eta})^2}{2m_1m_2\eta}\frac{1}{\eta} - i\frac{\vec{\kappa}\cdot\vec{\eta}}{4\eta^3m^2_1} - i\frac{\vec{\kappa}\cdot\vec{\eta}}{4\eta^3m^2_2}\Big)_{ordered} +\\
&-\frac{\vec{L}\cdot\vec{\sigma}_1}{4\eta^3m^2_1} - \frac{\vec{L}\cdot\vec{\sigma}_2}{4\eta^3m^2_2} - \frac{\vec{L}\cdot(\vec{\sigma}_1+\vec{\sigma}_2)}{2\eta^3m_1m_2} +\\
&+(-\frac{2\pi}{3}\delta(\vec{\eta})\vec{\sigma}_1\cdot\vec{\sigma}_2 + \frac{1}{4}\frac{\vec{\sigma}_1\cdot\vec{\sigma}_2}{\eta^3} - \frac{3}{4}\frac{\vec{\sigma}_1\cdot\hat{\eta}\vec{\sigma}_2\cdot\hat{\eta}}{\eta^3})\frac{1}{m_1m_2}\Big].
\end{aligned} \tag{8.16}$$

These terms, derived from the order $1/c^2$ relativistic corrections beyond Coulomb potential, are valid for hydrogen-like systems (either mass $\to\infty$), equal mass (positronium-like systems) or general unequal mass (muonium-like systems). This expression is the same as derived from the Bethe-Salpeter equation and Breit equation (see Refs. [14–16,18]) and by quantization of Wheeler-Feynman dynamics for particles with pseudo-classical spin [17].

IX. CONCLUSIONS.

In this paper we have shown that the effective lowest order Hamiltonian used for the theory of relativistic bound states with spin 1/2 constituents of arbitrary mass (muonium-, hydrogen- and positronium-like systems) may be derived as a *semi-classical* result implied by the Lienard-Wiechert solution in presence of positive-energy spinning particles in the rest-frame instant form of dynamics.

The result is non trivial in two respects:

1) At least to us it was not evident that the semi-classical treatment of the delay with Grassmann-valued electric charges with $Q_i^2 = 0$ and $Q_i Q_j \neq 0$ (regularizing the Coulomb energy and producing a unique semi-classical Lienard-Wiechert solution) could produce exactly a result derived from quantum field theory through the reduction of the Bethe-Salpeter equation.

2) It confirms the validity of the concept of spinning particle, with its semi-classical description of spin with Grassmann variables, as a semi-classical simulation of fermions and the relevance of the Foldy-Wouthuysen transformation in determining the effective action-at-a-distance interparticle potential.

Moreover, we obtain the full relativistic structure of the semi-classical approximation without any $1/c^2$ expansions. This could help in evaluating relativistic recoil effects.

We have already a semi-classical formulation for scalar quarks in terms of the Dirac observables in the non-Abelian radiation gauge in the rest-frame instant form of dynamics [7]. This has been obtained in the framework of weighted Sobolev spaces, where there is no Gribov ambiguity, in the case of a trivial SU(3)-principal bundle over each spacelike hypersurface. Having solved the Gauss laws, there is already a $1/r$ *Yang-Mills transverse-potential-dependent potential* in the final Hamiltonian. It acts between any pair of color charge densities, $\rho_a(\tau, \vec{\sigma}_1)$ and $\rho_b(\tau, \vec{\sigma}_2)$, [either quarks or localized charged Yang-Mills field (classical background of gluons)] and consists of three terms:

i) a Coulomb potential between $\rho_a(\tau, \vec{\sigma}_1)$ and $\rho_b(\tau, \vec{\sigma}_2)$;

ii) a potential in which $\rho_a(\tau, \vec{\sigma}_1)$ has a Coulomb interaction with an arbitrary *color-carrying center* located at $\vec{\sigma}_3$, while $\rho_b(\tau, \vec{\sigma}_2)$ interacts with the same center through a *Wilson line along the flat geodesics* $\vec{\sigma}_2 - \vec{\sigma}_3$ (plus the case $1 \leftrightarrow 2$): then one integrates over the location of the center;

iii) an interaction with *two centers* $\vec{\sigma}_3$ and $\vec{\sigma}_4$ (over whose location one integrates): $\rho_a(\tau, \vec{\sigma}_1)$ interacts with $\vec{\sigma}_3$ through a Wilson line, the same happens between $\rho_b(\tau, \vec{\sigma}_2)$ and $\vec{\sigma}_4$, and, finally, the two centers have a mutual Coulomb interaction.

As shown in Ref. [7] in the case of mesons (quark-antiquark system) the *semiclassical approximation of Grassmann-valued color charges* regularizes the potential and produces a *semiclassical asymptotic freedom*. However, due to the presence of the Wilson lines, it is not possible to check whether there is confinement. Neither baryons (3-quark system) nor the introduction of the spin along the lines of this paper have been studied till now.

The challenge now is to see whether with the technology of this paper it is possible to treat the non-Abelian case of the quark model, by using a perturbative treatment based on the iterative Lienard-Wiechert-type solutions developed in Ref. [33] adapted to this non-Abelian radiation gauge. Namely, whether it is possible to get a non-Abelian Darwin potential and to check if it is a confining potential.

APPENDIX A: EXPLICIT FORMS OF THE F AND G FUNCTIONS OF SECTION II.

To find the functions $F = F_2 - F_1$, $G = G_2 - G_1$ of Eqs.(2.18) we must preliminarly determine the functions Φ_n, Ψ_n of Eqs. (2.12), (2.15). In order to calculate Φ_n and Ψ_n from the definitions (2.12),(2.15), we observe that, since Eqs.(2.6), (2.7) imply

$$\begin{aligned} S &= S_{free} + \mathcal{O}(Q), \\ S_{free} &= 2i(\vec{p}\cdot\vec{\xi})\xi_5\theta(p), \end{aligned} \tag{A1}$$

then, Eq.(2.12) implies that for every n we have

$$\begin{aligned} 2i\xi_0\Phi_{2n} &= \{\Phi_{2n-1}\xi_5\xi_0, S_{free}\} \Rightarrow \Phi_{2n} = -i(\vec{p}\cdot\vec{\xi})\theta(p)\Phi_{2n-1}, \\ 2i\xi_5\xi_0\Phi_{2n+1} &= \{\Phi_{2n}\xi_0, S_{free}\} \Rightarrow \Phi_{2n+1} = -\{\Phi_{2n}, (\vec{p}\cdot\vec{\xi})\theta(p)\}. \end{aligned} \tag{A2}$$

Analogously for $n \geq 2$ Eq.(2.15) implies

$$\begin{aligned} 2i\xi_0\Psi_{2n} &= \{\Psi_{2n-1}\xi_5\xi_0, S_{free}\} \Rightarrow \Psi_{2n} = -i(\vec{p}\cdot\vec{\xi})\theta(p)\Psi_{2n-1}, \\ 2i\xi_5\xi_0\Psi_{2n+1} &= \{\Psi_{2n}\xi_0, S_{free}\} \Rightarrow \Psi_{2n+1} = -\{\Psi_{2n}, (\vec{p}\cdot\vec{\xi})\theta(p)\}. \end{aligned} \tag{A3}$$

From Eqs.(2.13), (2.15) we get at order Q $[\{QA_0\xi_0, S\} = 2iQ\Phi_1\xi_5\xi_0,\ \{p_0\xi_0, S\} = 2iQ\Psi_1\xi_5\xi_0]$

$$\begin{aligned} \Phi_1 &= \theta(p)\xi^k\frac{\partial A_0}{\partial x^k} + (\vec{p}\cdot\vec{\xi})\frac{\partial\theta(p)}{\partial p^u}\frac{\partial A_0}{\partial x^u}, \\ \Psi_1 &= -\theta(p)\vec{\xi}\cdot\frac{\partial\vec{A}}{\partial x^o} - 2\frac{d\theta(s)}{ds}|_{s=p^2}(\vec{p}\cdot\vec{\xi})\vec{p}\cdot\frac{\partial\vec{A}}{\partial x^o} + \frac{d\theta(s)}{ds}|_{s=p^2}(\vec{p}\cdot\vec{\xi})\left(-i\xi^h\xi^k\frac{\partial F_{hk}}{\partial x^o}\right). \end{aligned} \tag{A4}$$

Then, from $\{\{QA_o\xi_o, S\}, S\} = -4Q\Phi_2\xi_5\xi_o$, $\{\{p_o\xi_o, S\}, S\} = -4Q\Psi_2\xi_5\xi_o$ we obtain

$$\begin{aligned} \Phi_2 &= -i\theta^2(p)(\vec{p}\cdot\vec{\xi})\vec{\xi}\cdot\vec{\partial}A_0 \equiv f_{2,uv}\xi^u\xi^v, \\ \Psi_2 &= +i\theta^2(p)(\vec{p}\cdot\vec{\xi})\vec{\xi}\cdot\frac{\partial\vec{A}}{\partial x^o} \equiv g_{2,uv}\xi^u\xi^v, \end{aligned} \tag{A5}$$

where

$$\begin{aligned} f_{2,uv} &:= -i\theta^2(p)p^u\frac{\partial A_0}{\partial x^v}, \\ g_{2,uv} &:= +i\theta^2(p)p^u\frac{\partial A^v}{\partial x^o}. \end{aligned} \tag{A6}$$

1. Sum of Even Terms (Diagonal Terms).

We prove the following lemma:
Lemma 1

Lemma 1 *For $n \geq 1$:*

$$\Phi_{2n} = T_{(2n)}(p)(\vec{p} \cdot \xi)\vec{\xi} \cdot \vec{\partial} A_0(x), \tag{A7}$$

with

$$T_{(2n)}(p) = -i\frac{[p^2\theta^2(p)]^n}{p^2}. \tag{A8}$$

Proof: We first prove that, for $n \geq 1$ we have:

$$\Phi_{2n} = f_{2n,uv}\xi^u\xi^v. \tag{A9}$$

This equation is true for Φ_2; by assuming that it is true for Φ_{2n-2}, then by using Eq.(A2) we obtain:

$$\begin{aligned}\Phi_{2n-1} &= -\theta(p)p^k f_{2n-2,uv}\{\xi^u\xi^v, \xi^k\} - \{f_{2n-2,uv}, p^k\theta(p)\}\xi^u\xi^v\xi^k = \\ &= -i\theta(p)\left[p^u f_{2n-2,uv}\xi^v - \xi^u f_{2n-2,uv}p^v\right] + h_{uvk}\xi^u\xi^v\xi^k,\end{aligned} \tag{A10}$$

where

$$h_{uvk} = \{f_{2n-2,uv}, p^k\theta(p)\}. \tag{A11}$$

Using again Eq.(A2) and observing that $\xi^u\xi^v\xi^k\xi^h \equiv 0$, we get

$$\begin{aligned}\Phi_{2n} &= -i\theta(p)(\vec{p} \cdot \vec{\xi})\Phi_{2n-1} = \\ &= -\theta^2(p)(\vec{p} \cdot \vec{\xi})\left[p^u f_{2n-2,uv}\xi^v - \xi^u f_{2n-2,uv}p^v\right].\end{aligned} \tag{A12}$$

Thus Eq.(A9) is true with:

$$f_{2n,uv} = -\theta^2(p)p^u\left[p^k f_{2n-2,kv} - f_{2n-2,vk}p^k\right], \tag{A13}$$

and thus by induction it is true for all $n \geq 1$.

To complete the proof, we observe that the lemma is true for Φ_2 because:

$$f_{2,uv} = T_{(2)}(p)p^u\frac{\partial A_0}{\partial x^v}, \tag{A14}$$

with

$$T_{(2)}(p) = -i\theta^2(p). \tag{A15}$$

Again by induction, since we have

$$f_{2n-2,uv} = T_{(2n-2)}(p)p^u \frac{\partial A_0}{\partial x^v}, \tag{A16}$$

we can substitute this formula in Eq.(A9) and, using Eq.(A13), obtain

$$\Phi_{2n} = -\theta^2(p)p^u \left[p^k f_{2n-2,kv} - f_{2n-2,vk}p^k\right] \xi^u \xi^v =$$
$$= \theta^2(p)p^2 T_{(2n-2)}(p)(\vec{p}\cdot\xi)\vec{\xi}\cdot\vec{\partial}A_0(x). \tag{A17}$$

Finally, by using Eq.(A15) we get

$$T_{(2n)}(p) = \theta^2(p)p^2 T_{(2n-2)}(p) = -i\frac{[p^2\theta^2(p)]^n}{p^2}. \tag{A18}$$

Q.E.D.
In the same way we can to prove the following lemma:
Lemma 2

Lemma 2 *For* $n \geq 1$*:*

$$\Psi_{2n} = -T_{(2n)}(p)(\vec{p}\cdot\xi)\vec{\xi}\cdot\frac{\partial\vec{A}(x)}{\partial x^o}, \tag{A19}$$

with:

$$T_{(2n)}(p) = -i\frac{[p^2\theta^2(p)]^n}{p^2}. \tag{A20}$$

The proof is the same of lemma 1, with Φ_{2n} replaced by $(n \geq 1)$

$$\Psi_{2n} = g_{2n,uv}\xi^u\xi^v, \tag{A21}$$

and $f_{2n,uv}$ by

$$g_{2n,uv} = -T_{(2n)}(p)p^u\frac{\partial A^v(x)}{\partial x^o}. \tag{A22}$$

Eq.(A10) is replaced by

$$\Psi_{2n-1} = -i\theta(p)\left[p^u g_{2n-2,uv}\xi^v - \xi^u g_{2n-2,uv}p^v\right] + k_{uvk}\xi^u\xi^v\xi^k, \tag{A23}$$

with

$$k_{uvk} = \{g_{2n-2,uv}, p^k\theta(p)\}. \tag{A24}$$

Using the results of the two lemmas we find:

$$\Psi_{2n} - \Phi_{2n} = -T_{(2n)}(p)(\vec{p}\cdot\xi)\vec{\xi}\cdot\left[\frac{\partial\vec{A}(x)}{\partial x^0} + \vec{\partial}A_0(x)\right] = +T_{(2n)}(p)(\vec{p}\cdot\xi)\vec{\xi}\cdot\vec{E}(x). \tag{A25}$$

Then, we have the explicit form of the F function given in Eqs.(2.14,2.17,2.18)

$$
\begin{aligned}
F &= F_2 - F_1 = \\
&= A_0(x)\xi_0 + (\vec{p}\cdot\xi)\vec{\xi}\cdot\vec{E}(x)\sum_{k=1}^{\infty}\frac{(2i)^{2k}}{2k!}T_{(2k)}(p) = \\
&= A_0(x)\xi_0 - \frac{i}{p^2}(\vec{p}\cdot\xi)\vec{\xi}\cdot\vec{E}(x)\sum_{k=1}^{\infty}(-)^k\frac{2^{2k}p^{2k}\theta^{2k}(p)}{2k!} = \\
&= A_0(x)\xi_0 - \frac{i}{p^2}(\vec{p}\cdot\xi)\vec{\xi}\cdot\vec{E}(x)\left[\cos(2p\theta(p)) - 1\right] = \\
&= A_0(x)\xi_0 - i(\vec{p}\cdot\xi)(\vec{\xi}\cdot\vec{E}(x))\frac{m-\sqrt{m^2+\vec{p}^2}}{\vec{p}^2\sqrt{m^2+\vec{p}^2}} \\
&= A_0(x)\xi_0 - i(\vec{p}\cdot\xi)(\vec{\xi}\cdot\vec{E}(x))\frac{1}{(m+\sqrt{m^2+\vec{p}^2})\sqrt{m^2+\vec{p}^2}}.
\end{aligned} \tag{A26}
$$

2. Sum of Odd Terms (Skew-Diagonal Terms)

We calculate first G_1. In Eq.(A10) for Φ_{2n-1}

$$
\Phi_{2n-1} = -i\theta(p)\left[p^u f_{2n-2,uv}\xi^v - \xi^u f_{2n-2,uv}p^v\right] + h_{uvk}\xi^u\xi^v\xi^k, \tag{A27}
$$

we can substitute the expression for $f_{2n,uv}$

$$
f_{2n,uv} = T_{(2n)}(p)p^u\frac{\partial A_0}{\partial x^v}. \tag{A28}
$$

Since we have

$$
h_{uvk} = \{f_{2n-2,uv}, p^k\theta(p)\} = h_{ukv}, \tag{A29}
$$

then, by symmetry considerations, we get

$$
h_{uvk}\xi^u\xi^v\xi^k = 0. \tag{A30}
$$

We have

$$
\begin{aligned}
\Phi_{2n-1} &= -i\theta(p)\left[-i\frac{(p\theta(p))^{2n-2}}{p^2}\right]\Big((\vec{p}\cdot\vec{\partial}A_0)(\vec{p}\cdot\vec{\xi}) - p^2(\vec{\xi}\cdot\vec{\partial}A_0)\Big) = \\
&= -\frac{(p\theta(p))^{2n-1)}}{p^3}\Big((\vec{p}\cdot\vec{\partial}A_0)(\vec{p}\cdot\vec{\xi}) - p^2(\vec{\xi}\cdot\vec{\partial}A_0)\Big).
\end{aligned} \tag{A31}
$$

Then we obtain

$$
\begin{aligned}
G_1 &= 2i\phi_1 + \Big((\vec{p}\cdot\vec{\partial}A_0)(\vec{p}\cdot\vec{\xi}) - p^2(\vec{\xi}\cdot\vec{\partial}A_0)\Big)\sum_{k=1}^{\infty}(-)\frac{i^{2k+1}}{2k+1!}\frac{(2p\theta(p))^{2n-1}}{p^3} = \\
&= \frac{2ip\theta(p)}{p}(\vec{\xi}\cdot\vec{\partial}A_0) + 2i(\vec{p}\cdot\vec{\xi})\frac{\partial\theta(p)}{\partial p^u}\frac{\partial A_0}{\partial x^u} + \\
&+ i(\vec{\xi}\cdot\vec{\partial}A_0)\sum_{k=1}^{\infty}\frac{(-)^k}{2k+1!}\frac{(2p\theta(p))^{2n-1}}{p} +
\end{aligned}
$$

$$
\begin{aligned}
& - i(\vec{p}\cdot\vec{\xi})(\vec{p}\cdot\vec{\partial}A_0)\sum_{k=1}^{\infty}\frac{(-)^k}{2k+1!}\frac{(2p\theta(p))^{2n-1}}{p^3} = \\
& = i(\vec{\xi}\cdot\vec{\partial}A_0)\sum_{k=0}^{\infty}\frac{(-)^k}{2k+1!}\frac{(2p\theta(p))^{2n-1}}{p} + \\
& - i(\vec{p}\cdot\vec{\xi})(\vec{p}\cdot\vec{\partial}A_0)\sum_{k=0}^{\infty}\frac{(-)^k}{2k+1!}\frac{(2p\theta(p))^{2n-1}}{p^3} + \\
& + 2i(\vec{p}\cdot\vec{\xi})\frac{\partial\theta(p)}{\partial p^u}\frac{\partial A_0}{\partial x^u} + i\frac{2p\theta(p)}{p^3}(\vec{p}\cdot\vec{\xi})(\vec{p}\cdot\vec{\partial}A_0) = \\
& = +i\frac{(\vec{\xi}\cdot\vec{\partial}A_0)}{p}\sin[2p\theta(p)] - i\frac{(\vec{p}\cdot\vec{\xi})(\vec{p}\cdot\vec{\partial}A_0)}{p^3}\sin[2p\theta(p)] + \\
& + 2i(\vec{p}\cdot\vec{\xi})\frac{\partial\theta(p)}{\partial p^u}\frac{\partial A_0}{\partial x^u} + i\frac{2p\theta(p)}{p^3}(\vec{p}\cdot\vec{\xi})(\vec{p}\cdot\vec{\partial}A_0).
\end{aligned} \tag{A32}
$$

Now we calculate G_2. We have:

$$
\Psi_{2n-1} = -i\theta(p)\left[p^u g_{2n-2,uv}\xi^v - \xi^u g_{2n-2,uv}p^v\right] + k_{uvk}\xi^u\xi^v\xi^k, \tag{A33}
$$

and we can substitute the expression for $g_{2n,uv}$ into it

$$
g_{2n,uv} = -T_{(2n)}(p)p^u\frac{\partial A^v}{\partial x^o}. \tag{A34}
$$

With

$$
k_{uvk} = \{g_{2n-2,uv}, p^k\theta(p)\}, \tag{A35}
$$

we have in this case

$$
\begin{aligned}
k_{uvk}\xi^u\xi^v\xi^k &= -T_{(2n-2)}(p)p^u\left\{\frac{\partial A^v}{\partial x^0}, p^k\theta(p)\right\}\xi^u\xi^v\xi^k = \\
&= -T_{(2n-2)}(p)p^u\theta(p)\frac{\partial}{\partial x^0}\frac{\partial A^v}{\partial x^k}\xi^u\xi^v\xi^k = \\
&= \frac{-i}{2}\frac{(p\theta(p))^{2n-1}}{p^3}(\vec{p}\cdot\vec{\xi})\frac{\partial F_{hk}}{\partial x^0}\xi^h\xi^k.
\end{aligned} \tag{A36}
$$

We get

$$
\begin{aligned}
\Psi_{2n-1} &= +i\theta(p)\left[-i\frac{(p\theta(p))^{2n-2}}{p^2}\right]\left(\vec{p}\cdot\frac{\partial\vec{A}}{\partial x^o}(\vec{p}\cdot\vec{\xi}) - p^2\vec{\xi}\cdot\frac{\partial\vec{A}}{\partial x^o}\right) + \\
&- \frac{i}{2}\frac{(p\theta(p))^{2n-1}}{p^3}(\vec{p}\cdot\vec{\xi})\frac{\partial F_{hk}}{\partial x^o}\xi^h\xi^k = \\
&= +\frac{(p\theta(p))^{2n-1})}{p^3}\left(\vec{p}\cdot\frac{\partial\vec{A}}{\partial x^o}(\vec{p}\cdot\vec{\xi}) - p^2\vec{\xi}\cdot\frac{\partial\vec{A}}{\partial x^o}\right) + \\
&- \frac{i}{2}\frac{(p\theta(p))^{2n-1}}{p^3}(\vec{p}\cdot\vec{\xi})\frac{\partial F_{hk}}{\partial x^o}\xi^h\xi^k.
\end{aligned} \tag{A37}
$$

Then we have:

$$
\begin{aligned}
G_2 &= 2i\Psi_1 + \\
&+ \left[\left(\vec{p}\cdot\frac{\partial\vec{A}}{\partial x^o}(\vec{p}\cdot\vec{\xi}) - p^2\vec{\xi}\cdot\frac{\partial\vec{A}}{\partial x^o}\right) - \frac{i}{2}(\vec{p}\cdot\vec{\xi})\frac{\partial F_{hk}}{\partial x^o}\xi^h\xi^k\right]\sum_{k=1}^{\infty} i\frac{(-)^k}{p^3}\frac{(2p\theta(p))^{2k+1}}{2k+1!} = \\
&= -2i\theta(p)\vec{\xi}\cdot\frac{\partial\vec{A}}{\partial x^o} - 4i\frac{d\theta(s)}{ds}|_{s=p^2}(\vec{p}\cdot\vec{\xi})\vec{p}\cdot\frac{\partial\vec{A}}{\partial t} + \frac{d\theta(s)}{ds}|_{s=p^2}(\vec{p}\cdot\vec{\xi})\left(+2\xi^h\xi^k\frac{\partial F_{hk}}{\partial x^o}\right) + \\
&- \vec{\xi}\cdot\frac{\partial\vec{A}}{\partial x^o}\sum_{k=1}^{\infty} i\frac{(-)^k}{p}\frac{(2p\theta(p))^{2k+1}}{2k+1!} + \\
&+ \left(\vec{p}\cdot\frac{\partial\vec{A}}{\partial x^o}(\vec{p}\cdot\vec{\xi}) - \frac{i}{2}(\vec{p}\cdot\vec{\xi})\frac{\partial F_{hk}}{\partial x^o}\xi^h\xi^k\right)\sum_{k=1}^{\infty} i\frac{(-)^k}{p^3}\frac{(2p\theta(p))^{2k+1}}{2k+1!} \\
&= -\vec{\xi}\cdot\frac{\partial\vec{A}}{\partial x^o}\sum_{k=0}^{\infty} i\frac{(-)^k}{p}\frac{(2p\theta(p))^{2k+1}}{2k+1!} + \\
&+ \left(\vec{p}\cdot\frac{\partial\vec{A}}{\partial x^o}(\vec{p}\cdot\vec{\xi}) - \frac{i}{2}(\vec{p}\cdot\vec{\xi})\frac{\partial F_{hk}}{\partial x^o}\xi^h\xi^k\right)\sum_{k=0}^{\infty} i\frac{(-)^k}{p^3}\frac{(2p\theta(p))^{2k+1}}{2k+1!} + \\
&- i\frac{2p\theta(p)}{p}\left(\vec{p}\cdot\frac{\partial\vec{A}}{\partial x^o}(\vec{p}\cdot\vec{\xi}) - \frac{i}{2}(\vec{p}\cdot\vec{\xi})\frac{\partial F_{hk}}{\partial x^o}\xi^h\xi^k\right) + \\
&- 4i\frac{d\theta(s)}{ds}|_{s=p^2}(\vec{p}\cdot\vec{\xi})\vec{p}\cdot\frac{\partial\vec{A}}{\partial x^o} + \frac{d\theta(s)}{ds}|_{s=p^2}(\vec{p}\cdot\vec{\xi})\left(+2\xi^h\xi^k\frac{\partial F_{hk}}{\partial x^o}\right) = \\
&= -\vec{\xi}\cdot\frac{\partial\vec{A}}{\partial x^o}\frac{i}{p}\sin[2p\theta(p)] + \\
&+ \left(\vec{p}\cdot\frac{\partial\vec{A}}{\partial x^o}(\vec{p}\cdot\vec{\xi}) - \frac{i}{2}(\vec{p}\cdot\vec{\xi})\frac{\partial F_{hk}}{\partial x^o}\xi^h\xi^k\right)\frac{i}{p^3}\sin[2p\theta(p)] + \\
&- i\frac{2p\theta(p)}{p}\left(\vec{p}\cdot\frac{\partial\vec{A}}{\partial x^o}(\vec{p}\cdot\vec{\xi}) - \frac{i}{2}(\vec{p}\cdot\vec{\xi})\frac{\partial F_{hk}}{\partial x^o}\xi^h\xi^k\right) + \\
&- 4i\frac{d\theta(s)}{ds}|_{s=p^2}(\vec{p}\cdot\vec{\xi})\vec{p}\cdot\frac{\partial\vec{A}}{\partial x^o} + \frac{d\theta(s)}{ds}|_{s=p^2}(\vec{p}\cdot\vec{\xi})\left(+2\xi^h\xi^k\frac{\partial F_{hk}}{\partial x^o}\right).
\end{aligned}
\tag{A38}
$$

The complete expression for the function G is

$$
\begin{aligned}
G = G_2 - G_1 &= -(\vec{\xi}\cdot\vec{E})\frac{i}{p}\sin[2p\theta(p)] + \\
&+ \left((\vec{p}\cdot\vec{E})(\vec{p}\cdot\vec{\xi}) - \frac{i}{2}(\vec{p}\cdot\vec{\xi})\frac{\partial F_{hk}}{\partial x^o}\xi^h\xi^k\right)\frac{i}{p^3}\sin[2p\theta(p)] + \\
&+ i\frac{2p\theta(p)}{p}\left((\vec{p}\cdot\vec{E})(\vec{p}\cdot\vec{\xi}) - \frac{i}{2}(\vec{p}\cdot\vec{\xi})\frac{\partial F_{hk}}{\partial x^o}\xi^h\xi^k\right) + \\
&- 4i\frac{d\theta(s)}{ds}|_{s=p^2}(\vec{p}\cdot\vec{\xi})(\vec{p}\cdot\vec{E}) + \frac{d\theta(s)}{ds}|_{s=p^2}(\vec{p}\cdot\vec{\xi})\left(+2\xi^h\xi^k\frac{\partial F_{hk}}{\partial x^o}\right).
\end{aligned}
\tag{A39}
$$

APPENDIX B: SPACELIKE HYPERSURFACES.

1. Review of Their Properties.

Let us first review some preliminary results from Refs. [4,20] needed in the description of physical systems on spacelike hypersurfaces.

Let $\{\Sigma_\tau\}$ be a one-parameter family of spacelike hypersurfaces foliating Minkowski spacetime M^4 and giving a 3+1 decomposition of it. At fixed τ, let $z^\mu(\tau,\vec{\sigma})$ be the coordinates of the points on Σ_τ in M^4, $\{\vec{\sigma}\}$ a system of coordinates on Σ_τ. If $\sigma^{\check{A}} = (\sigma^\tau = \tau; \vec{\sigma} = \{\sigma^{\check{r}}\})$ [the notation $\check{A} = (\tau, \check{r})$ with $\check{r} = 1,2,3$ will be used; note that $\check{A} = \tau$ and $\check{A} = \check{r} = 1,2,3$ are Lorentz-scalar indices] and $\partial_{\check{A}} = \partial/\partial\sigma^{\check{A}}$, one can define the vierbeins

$$z^\mu_{\check{A}}(\tau,\vec{\sigma}) = \partial_{\check{A}} z^\mu(\tau,\vec{\sigma}), \qquad \partial_{\check{B}} z^\mu_{\check{A}} - \partial_{\check{A}} z^\mu_{\check{B}} = 0, \tag{B1}$$

so that the metric on Σ_τ is

$$\begin{aligned} &g_{\check{A}\check{B}}(\tau,\vec{\sigma}) = z^\mu_{\check{A}}(\tau,\vec{\sigma})\eta_{\mu\nu} z^\nu_{\check{B}}(\tau,\vec{\sigma}), \qquad g_{\tau\tau}(\tau,\vec{\sigma}) > 0,\\ &g(\tau,\vec{\sigma}) = -det\, ||\, g_{\check{A}\check{B}}(\tau,\vec{\sigma})\, || = (det\, ||\, z^\mu_{\check{A}}(\tau,\vec{\sigma})\, ||)^2,\\ &\gamma(\tau,\vec{\sigma}) = -det\, ||\, g_{\check{r}\check{s}}(\tau,\vec{\sigma})\, ||. \end{aligned} \tag{B2}$$

If $\gamma^{\check{r}\check{s}}(\tau,\vec{\sigma})$ is the inverse of the 3-metric $g_{\check{r}\check{s}}(\tau,\vec{\sigma})$ $[\gamma^{\check{r}\check{u}}(\tau,\vec{\sigma}) g_{\check{u}\check{s}}(\tau,\vec{\sigma}) = \delta^{\check{r}}_{\check{s}}]$, the inverse $g^{\check{A}\check{B}}(\tau,\vec{\sigma})$ of $g_{\check{A}\check{B}}(\tau,\vec{\sigma})$ $[g^{\check{A}\check{C}}(\tau,\vec{\sigma}) g_{\check{c}\check{b}}(\tau,\vec{\sigma}) = \delta^{\check{A}}_{\check{B}}]$ is given by

$$\begin{aligned} &g^{\tau\tau}(\tau,\vec{\sigma}) = \frac{\gamma(\tau,\vec{\sigma})}{g(\tau,\vec{\sigma})},\\ &g^{\tau\check{r}}(\tau,\vec{\sigma}) = -[\frac{\gamma}{g} g_{\tau\check{u}}\gamma^{\check{u}\check{r}}](\tau,\vec{\sigma}),\\ &g^{\check{r}\check{s}}(\tau,\vec{\sigma}) = \gamma^{\check{r}\check{s}}(\tau,\vec{\sigma}) + [\frac{\gamma}{g} g_{\tau\check{u}} g_{\tau\check{v}} \gamma^{\check{u}\check{r}}\gamma^{\check{v}\check{s}}](\tau,\vec{\sigma}), \end{aligned} \tag{B3}$$

so that $1 = g^{\tau\check{C}}(\tau,\vec{\sigma}) g_{\check{C}\tau}(\tau,\vec{\sigma})$ is equivalent to

$$\frac{g(\tau,\vec{\sigma})}{\gamma(\tau,\vec{\sigma})} = g_{\tau\tau}(\tau,\vec{\sigma}) - \gamma^{\check{r}\check{s}}(\tau,\vec{\sigma}) g_{\tau\check{r}}(\tau,\vec{\sigma}) g_{\tau\check{s}}(\tau,\vec{\sigma}). \tag{B4}$$

We have

$$z^\mu_\tau(\tau,\vec{\sigma}) = (\sqrt{\frac{g}{\gamma}}\, l^\mu + g_{\tau\check{r}}\gamma^{\check{r}\check{s}} z^\mu_{\check{s}})(\tau,\vec{\sigma}), \tag{B5}$$

and

$$\begin{aligned} \eta^{\mu\nu} &= z^\mu_{\check{A}}(\tau,\vec{\sigma}) g^{\check{A}\check{B}}(\tau,\vec{\sigma}) z^\nu_{\check{B}}(\tau,\vec{\sigma}) =\\ &= (l^\mu l^\nu + z^\mu_{\check{r}}\gamma^{\check{r}\check{s}} z^\nu_{\check{s}})(\tau,\vec{\sigma}), \end{aligned} \tag{B6}$$

where

$$l^\mu(\tau,\vec\sigma) = (\frac{1}{\sqrt{\gamma}}\epsilon^\mu{}_{\alpha\beta\gamma} z^\alpha_{\check 1} z^\beta_{\check 2} z^\gamma_{\check 3})(\tau,\vec\sigma),$$

$$l^2(\tau,\vec\sigma) = 1, \qquad l_\mu(\tau,\vec\sigma) z^\mu_{\check r}(\tau,\vec\sigma) = 0, \tag{B7}$$

is the unit (future pointing) normal to Σ_τ at $z^\mu(\tau,\vec\sigma)$.

For the volume element in Minkowski spacetime we have

$$d^4 z = z^\mu_\tau(\tau,\vec\sigma) d\tau d^3\Sigma_\mu = d\tau [z^\mu_\tau(\tau,\vec\sigma) l_\mu(\tau,\vec\sigma)] \sqrt{\gamma(\tau,\vec\sigma)} d^3\sigma =$$
$$= \sqrt{g(\tau,\vec\sigma)} d\tau d^3\sigma. \tag{B8}$$

Let us remark that according to the geometrical approach of Ref. [34],one can use Eq.(3.6) in the form $z^\mu_\tau(\tau,\vec\sigma) = N(\tau,\vec\sigma) l^\mu(\tau,\vec\sigma) + N^{\check r}(\tau,\vec\sigma) z^\mu_{\check r}(\tau,\vec\sigma)$, where $N = \sqrt{g/\gamma} = \sqrt{g_{\tau\tau} - \gamma^{\check r\check s} g_{\tau\check r} g_{\tau\check s}}$ and $N^{\check r} = g_{\tau\check s}\gamma^{\check s\check r}$ are the standard lapse and shift functions, so that $g_{\tau\tau} = N^2 + g_{\check r\check s} N^{\check r} N^{\check s}, g_{\tau\check r} = g_{\check r\check s} N^{\check s}, g^{\tau\tau} = N^{-2}, g^{\tau\check r} = -N^{\check r}/N^2, g^{\check r\check s} = \gamma^{\check r\check s} + \frac{N^{\check r} N^{\check s}}{N^2}$, $\frac{\partial}{\partial z^\mu_\tau} = l_\mu \frac{\partial}{\partial N} + z_{\check s\mu}\gamma^{\check s\check r}\frac{\partial}{\partial N^{\check r}}$, $d^4 z = N\sqrt{\gamma} d\tau d^3\sigma$.

The rest frame form of a timelike fourvector p^μ is $\overset{\circ}{p}{}^\mu = \eta\sqrt{p^2}(1;\vec 0) = \eta^{\mu o}\eta\sqrt{p^2}$, $\overset{\circ}{p}{}^2 = p^2$, where $\eta = sign\, p^o$. The standard Wigner boost transforming $\overset{\circ}{p}{}^\mu$ into p^μ is

$$\begin{aligned}
L^\mu{}_\nu(p,\overset{\circ}{p}) &= \epsilon^\mu_\nu(u(p)) = \\
&= \eta^\mu_\nu + 2\frac{p^\mu \overset{\circ}{p}_\nu}{p^2} - \frac{(p^\mu + \overset{\circ}{p}{}^\mu)(p_\nu + \overset{\circ}{p}_\nu)}{p\cdot \overset{\circ}{p} + p^2} = \\
&= \eta^\mu_\nu + 2u^\mu(p) u_\nu(\overset{\circ}{p}) - \frac{(u^\mu(p) + u^\mu(\overset{\circ}{p}))(u_\nu(p) + u_\nu(\overset{\circ}{p}))}{1 + u^o(p)},
\end{aligned}$$

$$\begin{aligned}
\nu = 0 \quad & \epsilon^\mu_o(u(p)) = u^\mu(p) = p^\mu/\eta\sqrt{p^2}, \\
\nu = r \quad & \epsilon^\mu_r(u(p)) = (-u_r(p); \delta^i_r - \frac{u^i(p) u_r(p)}{1 + u^o(p)}).
\end{aligned} \tag{B9}$$

The inverse of $L^\mu{}_\nu(p,\overset{\circ}{p})$ is $L^\mu{}_\nu(\overset{\circ}{p},p)$, the standard boost to the rest frame, defined by

$$L^\mu{}_\nu(\overset{\circ}{p},p) = L_\nu{}^\mu(p,\overset{\circ}{p}) = L^\mu{}_\nu(p,\overset{\circ}{p})|_{\vec p \to -\vec p}. \tag{B10}$$

Therefore, we can define the following vierbeins [27]

$$\begin{aligned}
&\epsilon^\mu_A(u(p)) = L^\mu{}_A(p,\overset{\circ}{p}), \\
&\epsilon^A_\mu(u(p)) = L^A{}_\mu(\overset{\circ}{p},p) = \eta^{AB}\eta_{\mu\nu}\epsilon^\nu_B(u(p)), \\
\\
&\epsilon^{\bar o}_\mu(u(p)) = \eta_{\mu\nu}\epsilon^\nu_o(u(p)) = u_\mu(p), \\
&\epsilon^r_\mu(u(p)) = -\delta^{rs}\eta_{\mu\nu}\epsilon^\nu_r(u(p)) = (\delta^{rs} u_s(p); \delta^r_j - \delta^{rs}\delta_{jh}\frac{u^h(p) u_s(p)}{1 + u^o(p)}), \\
&\epsilon^A_o(u(p)) = u_A(p),
\end{aligned} \tag{B11}$$

[27]The $\epsilon^\mu_r(u(p))$'s are also called polarization vectors; the indices r, s will be used for A=1,2,3 and $\bar o$ for A=0.

which satisfy

$$
\begin{aligned}
&\epsilon^A_\mu(u(p))\epsilon^\nu_A(u(p)) = \eta^\mu_\nu,\\
&\epsilon^A_\mu(u(p))\epsilon^\mu_B(u(p)) = \eta^A_B,\\
&\eta^{\mu\nu} = \epsilon^\mu_A(u(p))\eta^{AB}\epsilon^\nu_B(u(p)) = u^\mu(p)u^\nu(p) - \sum_{r=1}^3 \epsilon^\mu_r(u(p))\epsilon^\nu_r(u(p)),\\
&\eta_{AB} = \epsilon^\mu_A(u(p))\eta_{\mu\nu}\epsilon^\nu_B(u(p)),\\
&p_\alpha \frac{\partial}{\partial p_\alpha}\epsilon^\mu_A(u(p)) = p_\alpha \frac{\partial}{\partial p_\alpha}\epsilon^A_\mu(u(p)) = 0.
\end{aligned}
\tag{B12}
$$

The Wigner rotation corresponding to the Lorentz transformation Λ is

$$
\begin{aligned}
R^\mu{}_\nu(\Lambda, p) &= {[L(\overset{\circ}{p}, p)\Lambda^{-1}L(\Lambda p, \overset{\circ}{p})]^\mu}_\nu = \begin{pmatrix} 1 & 0 \\ 0 & R^i{}_j(\Lambda, p)\end{pmatrix},\\
R^i{}_j(\Lambda, p) &= (\Lambda^{-1})^i{}_j - \frac{(\Lambda^{-1})^i{}_o p_\beta (\Lambda^{-1})^\beta{}_j}{p_\rho(\Lambda^{-1})^\rho{}_o + \eta\sqrt{p^2}} - \\
&\quad - \frac{p^i}{p^o + \eta\sqrt{p^2}}[(\Lambda^{-1})^o{}_j - \frac{((\Lambda^{-1})^o{}_o - 1)p_\beta(\Lambda^{-1})^\beta{}_j}{p_\rho(\Lambda^{-1})^\rho{}_o + \eta\sqrt{p^2}}].
\end{aligned}
\tag{B13}
$$

The polarization vectors transform under the Poincaré transformations (a, Λ) in the following way

$$
\epsilon^\mu_r(u(\Lambda p)) = (R^{-1})_r{}^s\, \Lambda^\mu{}_\nu\, \epsilon^\nu_s(u(p)).
\tag{B14}
$$

2. General Form of the Constraints of Section IV on Arbitrary Spacelike Hypersurfaces.

Let us try to undo the canonical reduction of Section III and to find the general form of the modified constraints (4.4) outside the radiation gauge and outside the Wigner hyperplanes.

Since Eq.(3.35) implies

$$
\begin{aligned}
\pi^r_\perp(\tau, \vec\sigma) &= \pi^r(\tau, \vec\sigma) - \frac{\partial^r}{\triangle_\sigma}\Big[\Gamma(\tau, \vec\sigma) - \sum_{i=1}^N Q_i \delta^3(\vec\sigma - \vec\eta_i(\tau))\Big] \approx\\
&\approx \pi^r(\tau, \vec\sigma) + \frac{\partial^r}{\triangle_\sigma}\sum_{i=1}^N Q_i\delta^3(\vec\sigma - \vec\eta_i(\tau)) = \pi^r(\tau, \vec\sigma) + \sum_{i=1}^N Q_i c^r(\tau, \vec\sigma),\\
&\triangle = -\vec\partial^2, \qquad \triangle c(\vec\sigma) = \delta^3(\vec\sigma),\\
&\frac{1}{\triangle}\delta^3(\vec\sigma) = c(\vec\sigma) = \frac{-1}{4\pi|\vec\sigma|},\\
&c^r(\vec\sigma) = \partial^r c(\vec\sigma) = \frac{\partial^r}{\triangle}\delta^3(\vec\sigma) = \frac{\sigma^r}{4\pi|\vec\sigma|^3},\\
&\vec\partial\cdot\vec c(\vec\sigma) = -\delta^3(\vec\sigma),
\end{aligned}
\tag{B15}
$$

and since Eq.(3.39) allows to write the Coulomb potential in the following form

$$\begin{aligned} V_{ij} &= \frac{Q_iQ_j}{4\pi|\vec{\eta}_i(\tau)-\vec{\eta}_j(\tau)|} = -Q_iQ_j\int d^3\sigma \frac{\vec{\partial}}{\triangle_\sigma}\delta^3(\vec{\sigma}-\vec{\eta}_i(\tau))\cdot \frac{\vec{\partial}}{\triangle_\sigma}\delta^3(\vec{\sigma}-\vec{\eta}_j(\tau)) = \\ &= -Q_iQ_j\int d^3\sigma\, \vec{c}(\vec{\sigma}-\vec{\eta}_i(\tau))\cdot \vec{c}(\vec{\sigma}-\vec{\eta}_j(\tau)), \end{aligned} \tag{B16}$$

the general form of the modified constraint $\mathcal{H}^{'}(\tau)\approx 0$ on the Wigner hyperplane without the restriction to the radiation gauge is

$$\begin{aligned} \mathcal{H}^{'} &= \epsilon_s - \Big(\sum_{i=1}^N\sqrt{m_i^2 - iQ_i\xi_i^r(\tau)\xi_i^s(\tau)F_{rs}(\tau,\vec{\eta}_i(\tau)) + (\check{\vec{\kappa}}_i(\tau)-Q_i\check{\vec{A}}_\perp(\tau,\vec{\eta}_i(\tau)))^2} - \\ &- i\sum_{i=1}^N \frac{Q_i\check{\vec{\kappa}}_i(\tau)\cdot\vec{\xi}_i(\tau)\,\vec{\xi}_i(\tau)\cdot\check{\vec{\pi}}_\perp(\tau,\vec{\eta}_i(\tau))}{(m_i+\sqrt{m_i^2+\check{\vec{\kappa}}_i^2(\tau)})\sqrt{m_i^2+\check{\vec{\kappa}}_i^2(\tau)}} - \\ &- i\sum_{i\neq j}\frac{Q_iQ_j\check{\vec{\kappa}}_i(\tau)\cdot\vec{\xi}_i(\tau)}{(m_i+\sqrt{m_i^2+\check{\vec{\kappa}}_i^2(\tau)})\sqrt{m_i^2+\check{\vec{\kappa}}_i^2(\tau)}}\vec{\xi}_i(\tau)\cdot\frac{\partial}{\partial\vec{\eta}_i}\int d^3\sigma_1\,\vec{c}(\vec{\sigma}_1-\vec{\eta}_i(\tau))\cdot\vec{c}(\vec{\sigma}_1-\vec{\eta}_j(\tau)) + \\ &+ \int d^3\sigma\frac{1}{2}[\check{\vec{\pi}}_\perp^2+\check{\vec{B}}^2](\tau,\vec{\sigma})\Big)\approx 0. \end{aligned} \tag{B17}$$

Eq.(B16) suggests the following modification of the ten constraints (3.16) describing the isolated system on arbitrary hyperplanes (namely not on the Wigner hyperplanes)

$$\begin{aligned} \mathcal{H}_\mu(\tau) &= \int d^3\sigma\mathcal{H}_\mu(\tau,\vec{\sigma}) = \ p_{s_\mu} - b_{\mu\tau}\Big\{\frac{1}{2}\int d^3\sigma\Big(\vec{\pi}^2(\tau,\vec{\sigma})+\vec{B}^2(\tau,\vec{\sigma})\Big) + \\ &+ \sum_{i=1}^N\sqrt{m_i^2 - iQ_i\xi_i^r(\tau)\xi_i^s(\tau)F_{rs}(\tau,\vec{\eta}_i(\tau)) + +(\check{\vec{\kappa}}_i(\tau)-Q_i\check{\vec{A}}_\perp(\tau,\vec{\eta}_i(\tau)))^2} + \\ &+ \frac{1}{2}\sum_{i,j=1}^N\delta^3(\vec{\eta}_i(\tau)-\vec{\eta}_j(\tau))\cdot \\ &\cdot \frac{Q_i(\tau)Q_j(\tau)\xi_i^\gamma(\tau)\xi_i^\delta(\tau)}{\sqrt{m_i^2+\vec{\kappa}_i^2(\tau)}\sqrt{m_j^2+\vec{\kappa}_j^2(\tau)}}b_{\tau\gamma}\eta_{\delta\beta}\xi_j^\alpha(\tau)\xi_j^\beta(\tau)b_{\tau\alpha}\Big] + \\ &+ i\sum_{i=1}^N\frac{Q_i(\tau)\xi_i^\alpha(\tau)\xi_i^\beta(\tau)}{\sqrt{m_i^2+\vec{\kappa}_i^2(\tau)}}b_{\tau\alpha}b_{\check{s}\beta}(\tau)\pi^{\check{s}}(\tau,\vec{\eta}_i(\tau)) + \\ &- \frac{i}{2}\sum_{i=1}^N\frac{Q_i(\tau)\xi_i^\alpha(\tau)\xi_i^\beta(\tau)}{\sqrt{m_i^2+\vec{\kappa}_i^2(\tau)}}b_{\check{u}\alpha}(\tau)b_{\check{v}\beta}(\tau)F_{\check{u}\check{v}}(\tau,\vec{\eta}_i(\tau)) + \\ &+ \sum_{i=1}^N\int d^3\sigma\delta^3(\vec{\sigma}-\vec{\eta}_i(\tau)) \\ &\frac{iQ_i(\tau)\kappa_{i\check{r}}(\tau)\xi_i^\alpha(\tau)b_{\check{r}\alpha}(\tau)\xi_i^\beta(\tau)b_{\check{s}\beta}(\tau)\Big[\pi^{\check{s}}(\tau,\vec{\sigma})+\frac{\partial^{\check{s}}}{\triangle}\sum_{k=1}^N Q_k(\tau)\delta^3(\vec{\sigma}-\vec{\eta}_k(\tau))\Big]}{(m_i+\sqrt{m_i^2+\vec{\kappa}_i^2(\tau)})\sqrt{m_i^2+\vec{\kappa}_i^2(\tau)}} - \\ &- i\sum_{i\neq j}^{1..N}\frac{Q_i(\tau)Q_j(\tau)\kappa_{i\check{r}}(\tau)\xi_i^\alpha(\tau)b_{\check{r}\alpha}(\tau)}{(m_i+\sqrt{m_i^2+\vec{\kappa}_i^2(\tau)})\sqrt{m_i^2+\vec{\kappa}_i^2(\tau)}} \end{aligned}$$

$$\xi^{\beta}_i(\tau) b_{\check{s}\beta}(\tau) \int d^3\sigma \Big[\frac{\partial}{\partial \sigma^{\check{s}}} \frac{\vec{\partial}}{\triangle} \delta(\vec{\sigma}-\vec{\eta}_i(\tau))\Big] \cdot \frac{\vec{\partial}}{\triangle}\delta^3(\vec{\sigma}-\vec{\eta}_j(\tau))\Big\} +$$

$$+ b_{\check{r}\mu}(\tau)\Big\{ \int d^3\sigma [\vec{\pi}\times \vec{B}]_{\check{r}}(\tau,\vec{\sigma}) + \sum_{i=1}^N [\kappa_{i\check{r}}(\tau) - Q_i(\tau) A_{\check{r}}(\tau,\vec{\eta}_i(\tau))]\Big\} \approx 0,$$

$$\begin{aligned}
\mathcal{H}^{\mu\nu}(\tau) &= b^{\mu}_{\check{r}}(\tau) \int d^3\sigma \sigma^{\check{r}} \mathcal{H}^{\nu}(\tau,\vec{\sigma}) - b^{\nu}_{\check{r}}(\tau) \int d^3\sigma \sigma^{\check{r}} \mathcal{H}^{\mu}(\tau,\vec{\sigma}) = \\
&= S^{\mu\nu}_s(\tau) - (b^{\mu}_{\check{r}}(\tau) b^{\nu}_{\tau} - b^{\nu}_{\check{r}}(\tau) b^{\mu}_{\tau})\Big\{\frac{1}{2}\int d^3\sigma \sigma^{\check{r}} \Big(\vec{\pi}^2(\tau,\vec{\sigma}) + \vec{B}^2(\tau,\vec{\sigma})\Big) + \\
&+ \frac{1}{2}\sum_{i,j=1}^N \delta^3(\vec{\eta}_i(\tau)-\vec{\eta}_j(\tau)) \cdot \\
&\cdot \frac{\eta^{\check{r}}_i Q_i(\tau) Q_j(\tau) \xi^{\gamma}_i(\tau)\xi^{\delta}_i(\tau)}{\sqrt{m_i^2+\vec{\kappa}_i^2(\tau)}\sqrt{m_j^2+\vec{\kappa}_j^2(\tau)}} b_{\tau\gamma}\eta_{\delta\beta}\xi^{\alpha}_j(\tau)\xi^{\beta}_j(\tau) b_{\tau\alpha} + \\
&+ \sum_{i=1}^N \eta^{\check{r}}_i \sqrt{m_i^2 + [\vec{\kappa}_i(\tau) - Q_i(\tau)\vec{A}(\tau,\vec{\eta}_i(\tau))]^2} + \\
&+ i \sum_{i=1}^N \eta^{\check{r}}_i(\tau) \frac{Q_i(\tau)\xi^{\alpha}_i(\tau)\xi^{\beta}_i(\tau)}{\sqrt{m_i^2+\vec{\kappa}_i^2(\tau)}} b_{\tau\alpha} b_{\check{s}\beta}(\tau) \pi^{\check{s}}(\tau,\vec{\eta}_i(\tau)) + \\
&- \frac{i}{2}\sum_{i=1}^N \eta^{\check{r}}_i(\tau) \frac{Q_i(\tau)\xi^{\alpha}_i(\tau)\xi^{\beta}_i(\tau)}{\sqrt{m_i^2+\vec{\kappa}_i^2(\tau)}} b_{\check{u}\alpha}(\tau) b_{\check{v}\beta}(\tau) F_{\check{u}\check{v}}(\tau,\vec{\eta}_i(\tau)) + \\
&+ i \sum_{i=1}^N \eta^{\check{r}}_i(\tau) \\
&\frac{i Q_i(\tau)\kappa_{i\check{r}}(\tau)\xi^{\alpha}_i(\tau) b_{\check{r}\alpha}(\tau)\xi^{\beta}_i(\tau) b_{\check{s}\beta}(\tau)\Big[\pi^{\check{s}}(\tau,\vec{\sigma}) + \frac{\partial^{\check{s}}}{\triangle}\sum_{k=1}^N Q_k(\tau)\delta^3(\vec{\sigma}-\vec{\eta}_k(\tau))\Big]}{(m_i + \sqrt{m_i^2+\vec{\kappa}_i^2})\sqrt{m_i^2+\vec{\kappa}_i^2}} - \\
&- i \sum_{i\neq j}^{1..N} \frac{Q_i(\tau)Q_j(\tau)\kappa_{i\check{r}}(\tau)\xi^{\alpha}_i(\tau) b_{\check{r}\alpha}(\tau)}{(m_i + \sqrt{m_i^2+\vec{\kappa}_i^2})\sqrt{m_i^2+\vec{\kappa}_i^2}} \\
&\xi^{\beta}_i(\tau) b_{\check{s}\beta}(\tau) \int d^3\sigma \sigma^{\check{r}} \Big[\frac{\partial}{\partial \sigma^{\check{s}}} \frac{\vec{\partial}}{\triangle}\delta(\vec{\sigma}-\vec{\eta}_i(\tau))\Big] \cdot \frac{\vec{\partial}}{\triangle}\delta^3(\vec{\sigma}-\vec{\eta}_j(\tau))\Big\} + \\
&+ (b^{\mu}_{\check{r}}(\tau) b^{\nu}_{\check{s}}(\tau) - b^{\nu}_{\check{r}}(\tau) b^{\mu}_{\check{s}}(\tau))\Big\{\int \; d^3\sigma\sigma^{\check{r}} [\vec{\pi}\times\vec{B}]_{\check{s}}(\tau,\vec{\sigma}) + \\
&+ \sum_{i=1}^N \eta^{\check{r}}_i(\tau)[\kappa_{i\check{s}}(\tau) - Q_i(\tau) A_{\check{s}}(\tau,\vec{\eta}_i(\tau))]\Big\} \approx 0.
\end{aligned} \qquad \text{(B18)}$$

However, there could be extra terms containing $\xi^{\mu}_i b_{\tau\mu}$ [like the spin-spin and spin-electric field terms in Eq.(3.16)], which vanish on the Wigner hyperplanes. To check whether they are needed, one should verify that the Poisson brackets of the ten modified constraints among themselves close on the constraints as it happens with Eq.(3.16). We are not going to do this check here, since we are only interested in the rest-frame instant form of dynamics

Again modulo terms in $\xi^{\mu}_i z_{\tau\mu}(\tau,\vec{\sigma})$ the modification of the original constraints of Eq.(3.3) on arbitrary spacelike hypersurfaces is

$$
\begin{aligned}
\mathcal{H}_\mu(\tau,\vec{\sigma}) &= \rho_\mu(\tau,\vec{\sigma}) - l_\mu(\tau,\vec{\sigma})\Big[-\frac{1}{2\sqrt{\gamma(\tau,\vec{\sigma})}}\pi^{\check{r}}(\tau,\vec{\sigma})g_{\check{r}\check{s}}(\tau,\vec{\sigma})\pi^s(\tau,\vec{\sigma}) + \\
&+ \frac{\sqrt{\gamma(\tau,\vec{\sigma})}}{4}\gamma^{\check{r}\check{s}}(\tau,\vec{\sigma})\gamma^{\check{u}\check{v}}(\tau,\vec{\sigma})F_{\check{r}\check{u}}(\tau,\vec{\sigma})F_{\check{s}\check{v}}(\tau,\vec{\sigma}) + \\
&+ \sum_{i=1}^N \delta^3(\vec{\sigma}-\vec{\eta}_i(\tau))\cdot \\
&\cdot \sqrt{m_i^2 - \gamma^{\check{r}\check{s}}(\tau,\vec{\sigma})(\kappa_{i\check{r}}(\tau) - Q_i(\tau)A_{\check{r}}(\tau,\vec{\sigma}))(\kappa_{i\check{s}}(\tau) - Q_i(\tau)A_{\check{s}}(\tau,\vec{\sigma}))} + \\
&+ \frac{1}{2\sqrt{\gamma(\tau,\vec{\sigma})}}\sum_{i,j=1}^N \delta^3(\vec{\sigma}-\vec{\eta}_i(\tau))\delta^3(\vec{\sigma}-\vec{\eta}_j(\tau))\cdot \\
&\cdot \frac{Q_i(\tau)Q_j(\tau)\xi_i^\gamma(\tau)\xi_i^\delta(\tau)l_\gamma(\tau,\vec{\sigma})\eta_{\delta\beta}\xi_j^\alpha(\tau)\xi_j^\beta(\tau)l_\alpha(\tau,\vec{\sigma})}{\sqrt{m_i^2 - \gamma^{\check{r}\check{s}}(\tau,\vec{\sigma})\kappa_{i\check{r}}(\tau)\kappa_{i\check{s}}(\tau)}\sqrt{m_j^2 - \gamma^{\check{r}\check{s}}(\tau,\vec{\sigma})\kappa_{j\check{r}}(\tau)\kappa_{j\check{s}}(\tau)}} + \\
&+ \frac{i}{\sqrt{\gamma(\tau,\vec{\sigma})}}\sum_{i=1}^N \delta^3(\vec{\sigma}-\vec{\eta}_i(\tau))\cdot \\
&\cdot \frac{Q_i(\tau)\xi_i^\alpha(\tau)\xi_i^\beta(\tau)}{\sqrt{m_i^2 - \gamma^{\check{r}\check{s}}(\tau,\vec{\sigma})\kappa_{i\check{r}}(\tau)\kappa_{i\check{s}}(\tau)}} l_\alpha(\tau,\vec{\sigma})z_{\check{s}\beta}(\tau,\vec{\sigma})\pi^{\check{s}}(\tau,\vec{\sigma}) - \\
&- \frac{i}{2}\sum_{i=1}^N \delta^3(\vec{\sigma}-\vec{\eta}_i(\tau))\frac{Q_i(\tau)\xi_i^\alpha(\tau)\xi_i^\beta(\tau)}{\sqrt{m_i^2 - \gamma^{\check{r}\check{s}}(\tau,\vec{\sigma})\kappa_{i\check{r}}(\tau)\kappa_{i\check{s}}(\tau)}}\cdot \\
&\cdot z_{\check{u}\alpha}(\tau,\vec{\sigma})z_{\check{v}\beta}(\tau,\vec{\sigma})\gamma^{\check{r}\check{u}}(\tau,\vec{\sigma})\gamma^{\check{s}\check{v}}(\tau,\vec{\sigma})F_{\check{r}\check{s}}(\tau,\vec{\sigma}) + \\
&+ i\delta^3(\vec{\sigma}-\vec{\eta}_i(\tau)) \\
&\frac{Q_i(\tau)\kappa_{i\check{r}}(\tau)\xi_i^\alpha(\tau)z_{\check{r}\alpha}(\tau,\vec{\sigma})\xi_i^\beta(\tau)z_{\check{s}\beta}(\tau,\vec{\sigma})\Big[\pi^{\check{s}}(\tau,\vec{\sigma}) + \frac{\partial^{\check{s}}}{\triangle}\sum_{k=1}^N Q_k(\tau)\delta^3(\vec{\sigma}-\vec{\eta}_k(\tau))\Big]}{(m_i + \sqrt{m_i^2 - \gamma^{\check{u}\check{v}}(\tau,\vec{\sigma})\kappa_{i\check{u}}(\tau)\kappa_{i\check{v}}(\tau)})\sqrt{m_i^2 - \gamma^{\check{u}\check{v}}(\tau,\vec{\sigma})\kappa_{i\check{u}}(\tau)\kappa_{i\check{v}}(\tau)}} + \\
&+ i\sum_{i\neq j}^{1..N} Q_iQ_j\gamma^{\check{u}\check{v}}(\tau,\vec{\sigma}) \\
&\frac{\partial_{\check{u}}}{\triangle}\Big[\frac{\kappa_{i\check{r}}(\tau)\xi_i^\alpha(\tau)z_{\check{r}\alpha}(\tau,\vec{\sigma})\xi_i^\beta(\tau)z_{\check{s}\beta}(\tau,\vec{\sigma})}{(m_i + \sqrt{m_i^2 - \gamma^{\check{u}\check{v}}(\tau,\vec{\sigma})\kappa_{i\check{u}}(\tau)\kappa_{i\check{v}}(\tau)})\sqrt{m_i^2 - \gamma^{\check{u}\check{v}}(\tau,\vec{\sigma})\kappa_{i\check{u}}(\tau)\kappa_{i\check{v}}(\tau)}} \\
&\frac{\partial}{\partial\sigma^{\check{s}}}\delta^3(\vec{\sigma}-\vec{\eta}_i(\tau))\Big]\frac{\partial_{\check{v}}}{\triangle}\delta^3(\vec{\sigma}-\vec{\eta}_j(\tau))\Big] - \\
&- \gamma^{\check{r}\check{s}}(\tau,\vec{\sigma})z_{\check{s}\mu}(\tau,\vec{\sigma})\Big[F_{\check{r}\check{u}}(\tau,\vec{\sigma})\pi^{\check{u}}(\tau,\vec{\sigma}) + \\
&+ \sum_{i=1}^N \delta^3(\vec{\sigma}-\vec{\eta}_i(\tau))(\kappa_{\check{r}i} - Q_i(\tau)A_{\check{r}}(\tau,\vec{\sigma}))\Big] \approx 0. \qquad \text{(B19)}
\end{aligned}
$$

APPENDIX C: COMPUTATION OF THE FIELD ENERGY AND MOMENTUM INTEGRALS OF SECTION VI.

Here we carry out the details in the computation of the field energy and momentum for the case $N = 2$. The general N results obtained in the text are an immediate generalization. From Eq.(6.41) and Eq.(6.43) we find that

$$
\begin{aligned}
\vec{E}^2_{\perp S}(\tau,\sigma) = & \frac{Q_1Q_2}{8\pi^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(2n)!(2m)!}\,\vec{\mathbf{V}}_1\cdot\vec{\mathbf{V}}_2\times \\
& \Big[(\dot{\vec{\eta}}_1\cdot\vec{\partial}_\sigma)^{2m+1}|\vec{\sigma}-\vec{\eta}_1|^{2m-1}\Big]\Big[(\dot{\vec{\eta}}_2\cdot\vec{\partial}_\sigma)^{2n+1}|\vec{\sigma}-\vec{\eta}_2|^{2n-1}\Big] - \\
& -\frac{Q_1Q_2}{8\pi^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(2n+2)!(2m)!}(\vec{\mathbf{V}}_1\cdot\vec{\partial}_\sigma)(\vec{\mathbf{U}}_2\cdot\vec{\partial}_\sigma) \\
& \Big[(\dot{\vec{\eta}}_1\cdot\vec{\partial}_\sigma)^{2m+1}|\vec{\sigma}-\vec{\eta}_1|^{2m-1}\Big]\Big[(\dot{\vec{\eta}}_2\cdot\vec{\partial}_\sigma)^{2n+1}|\vec{\sigma}-\vec{\eta}_2|^{2n+1}\Big] - \\
& -\frac{Q_1Q_2}{8\pi^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(2m+2)!(2n)!}(\vec{\mathbf{V}}_2\cdot\vec{\partial}_\sigma)(\vec{\mathbf{U}}_1\cdot\vec{\partial}_\sigma) \\
& \Big[(\dot{\vec{\eta}}_2\cdot\vec{\partial}_\sigma)^{2n+1})|\vec{\sigma}-\vec{\eta}_2|^{2n-1}\Big]\Big[(\dot{\vec{\eta}}_1\cdot\vec{\partial}_\sigma)^{2m+1})|\vec{\sigma}-\vec{\eta}_1|^{2m+1}\Big] + \\
& +\frac{Q_1Q_2}{8\pi^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(2m+2)!(2n+2)!}(\vec{\mathbf{U}}_2\cdot\vec{\partial}_\sigma)(\vec{\mathbf{U}}_1\cdot\vec{\partial}_\sigma) \\
& \Big[\vec{\partial}_\sigma(\dot{\vec{\eta}}_2\cdot\vec{\partial}_\sigma)^{2n+1})|\vec{\sigma}-\vec{\eta}_2|^{2n+1}\Big]\cdot\Big[\vec{\partial}_\sigma(\dot{\vec{\eta}}_1\cdot\vec{\partial}_\sigma)^{2m+1})|\vec{\sigma}-\vec{\eta}_1|^{2m+1}\Big],
\end{aligned}
\tag{C1}
$$

$$
\begin{aligned}
\vec{B}^2_{S}(\tau,\vec{\sigma}) = & \frac{Q_1Q_2}{8\pi^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(2n)!(2m)!}(\vec{\mathbf{V}}_1\cdot\vec{\mathbf{V}}_2) \\
& \Big[(\vec{\partial}_\sigma(\dot{\vec{\eta}}_1\cdot\vec{\partial}_\sigma)^{2m})|\vec{\sigma}-\vec{\eta}_1|^{2m-1}\Big]\cdot\Big[(\vec{\partial}_\sigma(\dot{\vec{\eta}}_2\cdot\vec{\partial}_\sigma)^{2n})|\vec{\sigma}-\vec{\eta}_2|^{2n-1}\Big] - \\
& -\frac{Q_1Q_2}{8\pi^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(2n)!(2m)!}(\vec{\mathbf{V}}_1\cdot\vec{\partial}_\sigma)(\vec{\mathbf{V}}_2\cdot\vec{\partial}_\sigma) \\
& \Big[(\dot{\vec{\eta}}_1\cdot\vec{\partial}_\sigma)^{2m})|\vec{\sigma}-\vec{\eta}_1|^{2m-1}\Big]\cdot\Big[(\dot{\vec{\eta}}_2\cdot\vec{\partial}_\sigma)^{2n})|\vec{\sigma}-\vec{\eta}_2|^{2n-1}\Big],
\end{aligned}
\tag{C2}
$$

$$
\begin{aligned}
& (\vec{E}_{\perp S}(\tau,\vec{\sigma})\times\vec{B}_S(\tau,\vec{\sigma}))_k = \\
= & \frac{Q_1Q_2}{16\pi^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(2n)!(2m)!}\,\vec{\mathbf{V}}_1\cdot\vec{\mathbf{V}}_2\times \\
& \Big[(\dot{\vec{\eta}}_1\cdot\vec{\partial}_\sigma)^{2m+1}|\vec{\sigma}-\vec{\eta}_1|^{2m-1}\Big]\Big[\vec{\partial}_\sigma(\dot{\vec{\eta}}_2\cdot\vec{\partial}_\sigma)^{2n}|\vec{\sigma}-\vec{\eta}_2|^{2n-1}\Big] - \\
& -\frac{Q_1Q_2}{16\pi^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(2n)!(2m)!}\,\vec{\mathbf{V}}_2\times \\
& \Big[(\vec{\mathbf{V}}_1\cdot\vec{\partial}_\sigma)(\dot{\vec{\eta}}_2\cdot\vec{\partial}_\sigma)^{2n}|\vec{\sigma}-\vec{\eta}_2|^{2n-1}\Big]\Big[(\dot{\vec{\eta}}_1\cdot\vec{\partial}_\sigma)^{2m+1}|\vec{\sigma}-\vec{\eta}_1|^{2m-1}\Big] - \\
& -\frac{Q_1Q_2}{16\pi^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(2n)!(2m+2)!}(\vec{\mathbf{V}}_2\cdot\vec{\partial}_\sigma)(\vec{\mathbf{U}}_1\cdot\vec{\partial}_\sigma)\times \\
& \Big[\vec{\partial}_\sigma(\dot{\vec{\eta}}_2\cdot\vec{\partial}_\sigma)^{2n}|\vec{\sigma}-\vec{\eta}_2|^{2n-1}\Big]\Big[(\dot{\vec{\eta}}_1\cdot\vec{\partial}_\sigma)^{2m+1}|\vec{\sigma}-\vec{\eta}_1|^{2m+1}\Big] +
\end{aligned}
$$

$$+\frac{Q_1Q_2}{16\pi^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(2n)!(2m+2)!}\vec{\mathbf{V}}_2\times$$
$$\Big[\vec{\partial}_\sigma(\vec{\mathbf{U}}_1\cdot\vec{\partial}_\sigma)(\dot{\vec{\eta}}_1\cdot\vec{\partial}_\sigma)^{2m+1}|\vec{\sigma}-\vec{\eta}_1|^{2m+1}\Big]\cdot\Big[\vec{\partial}_\sigma(\dot{\vec{\eta}}_2\cdot\vec{\partial}_\sigma)^{2n}|\vec{\sigma}-\vec{\eta}_2|^{2n-1}\Big]+$$
$$+(1\longleftrightarrow 2). \tag{C3}$$

Our aim here is to compute $\frac{1}{2}\int d^3\sigma(\vec{E}^2_{\perp S}+\vec{B}^2_S)(\tau,\vec{\sigma})$ and $\int d^3\sigma(\vec{E}_{\perp S}\times\vec{B}_S)(\tau,\vec{\sigma})$. Essentially these integrals were computed in detail in Appendix B of [5] for scalar particles which correspond here to the result obtained by replacing $\vec{\mathbf{V}}_i,\vec{\mathbf{U}}_i\rightarrow\dot{\vec{\eta}}_i$. Following steps in that appendix similar to ones that lead to Eqs.(6.4) and (6.5) in [5] gives the result Eq.(6.46), (6.47)

APPENDIX D: COMPARISON OF $U_1(\tau)$ WITH $U(\tau)$.

Let us compare the terms (7.7), (7.8) and (7.9) of the potential $U_1(\tau)$ with the terms of the potential $U(\tau)$ of Eq.(6.46) and let us try to combine them together so that, after the addition of $V_{DSO}(\tau)$ of Eq. (7.5), we can find the expression of the Darwin and spin-dependent terms of the potential $V_{DS}(\tau)$ of Eq.(7.2). We rewrite $U(\tau)$ of Eq.(6.46) in the following form

$$
\begin{aligned}
U(\tau) &= \frac{1}{2}\int d^3\sigma (\vec{E}^2_{\perp S} + \vec{B}^2_S)(\tau,\vec{\sigma}) = \\
&= \sum_{i<j}^{1..N} \frac{Q_iQ_j}{4\pi} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Big[\vec{\mathbf{V}}_i\cdot\vec{\mathbf{V}}_j (\frac{\vec{\kappa}_i}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\cdot\vec{\partial}_{ij})^{2m+1}(\frac{\vec{\kappa}_j}{\sqrt{m_j^2+\vec{\kappa}_j{}^2}}\cdot\vec{\partial}_{ij})^{2n+1}\frac{\eta_{ij}^{2n+2m+1}}{(2n+2m+2)!} - \\
&- (\vec{\mathbf{V}}_i\cdot\vec{\partial}_{ij})(\vec{\mathbf{U}}_j\cdot\vec{\partial}_{ij})(\frac{\vec{\kappa}_i}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\cdot\vec{\partial}_{ij})^{2m+1}(\frac{\vec{\kappa}_j}{\sqrt{m_j^2+\vec{\kappa}_j{}^2}}\cdot\vec{\partial}_{ij})^{2n+1}\frac{\eta_{ij}^{2n+2m+3}}{(2n+2m+4)!} - \\
&\quad -(\vec{\mathbf{U}}_i\cdot\vec{\partial}_{ij})(\vec{\mathbf{V}}_j\cdot\vec{\partial}_{ij})(\frac{\vec{\kappa}_i}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\cdot\vec{\partial}_{ij})^{2m+1}(\frac{\vec{\kappa}_j}{\sqrt{m_j^2+\vec{\kappa}_j{}^2}}\cdot\vec{\partial}_{ij})^{2n+1}\frac{\eta_{ij}^{2n+2m+3}}{(2n+2m+4)!} + \\
&\quad +(\vec{\mathbf{U}}_i\cdot\vec{\partial}_{ij})(\vec{\mathbf{U}}_j\cdot\vec{\partial}_{ij})(\frac{\vec{\kappa}_i}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\cdot\vec{\partial}_{ij})^{2m+1}(\frac{\vec{\kappa}_j}{\sqrt{m_j^2+\vec{\kappa}_j{}^2}}\cdot\vec{\partial}_{ij})^{2n+1}\frac{\eta_{ij}^{2n+2m+3}}{(2n+2m+4)!} + \\
&+ \vec{\mathbf{V}}_i\cdot\vec{\mathbf{V}}_j (\frac{\vec{\kappa}_i}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\cdot\vec{\partial}_{ij})^{2m}(\frac{\vec{\kappa}_j}{\sqrt{m_j^2+\vec{\kappa}_j{}^2}}\cdot\vec{\partial}_{ij})^{2n}\frac{\eta_{ij}^{2n+2m-1}}{(2n+2m)!} - \\
&- (\vec{\mathbf{V}}_i\cdot\vec{\partial}_{ij})(\vec{\mathbf{V}}_j\cdot\vec{\partial}_{ij})(\frac{\vec{\kappa}_i}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\cdot\vec{\partial}_{ij})^{2m}(\frac{\vec{\kappa}_j}{\sqrt{m_j^2+\vec{\kappa}_j{}^2}}\cdot\vec{\partial}_{ij})^{2n}\frac{\eta_{ij}^{2n+2m+1}}{(2n+2m+2)!}\Big],
\end{aligned} \tag{D1}
$$

where we have rewritten the operators $\vec{\mathbf{V}}_i$ of Eq.(5.13) and $\vec{\mathbf{U}}_i$ of Eq.(6.42)in the form [$\vec{\mathbf{k}}_i$ and $\vec{\mathbf{l}}_i$ are operators]

$$
\begin{aligned}
\vec{\mathbf{V}}_i &= \frac{\vec{\kappa}_i}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}} - \frac{i\vec{\xi}_i\vec{\xi}_i\cdot\vec{\partial}_{ij}}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}} + \frac{i\vec{\kappa}_i\cdot\vec{\partial}_{ij}\vec{\kappa}_i\cdot\vec{\xi}_i\vec{\xi}_i}{(m_i+\sqrt{m_i^2+\vec{\kappa}_i{}^2})(m_i^2+\vec{\kappa}_i{}^2)} := \frac{\vec{\mathbf{k}}_i}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}},\\
\vec{\mathbf{U}}_i &= \frac{\vec{\kappa}_i}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}} + \frac{i\vec{\kappa}_i\cdot\vec{\partial}_{ij}\vec{\kappa}_i\cdot\vec{\xi}_i\vec{\xi}_i}{(m_i+\sqrt{m_i^2+\vec{\kappa}_i{}^2})(m_i^2+\vec{\kappa}_i{}^2)} := \frac{\vec{\mathbf{l}}_i}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}.
\end{aligned} \tag{D2}
$$

We consider the last two lines in the expression for $U(\tau)$ given in Eq.(6.46)

$$
\begin{aligned}
&\sum_{i<j}^{1..N}\frac{Q_iQ_j}{4\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Big[\frac{\vec{\mathbf{k}}_{\mathbf{i}}}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\cdot\frac{\vec{\mathbf{k}}_{\mathbf{j}}}{\sqrt{m_j^2+\vec{\kappa}_j{}^2}}\\
&(\frac{\vec{\kappa}_i}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\cdot\vec{\partial}_{ij})^{2m}(\frac{\vec{\kappa}_j}{\sqrt{m_j^2+\vec{\kappa}_j{}^2}}\cdot\vec{\partial}_{ij})^{2n}\frac{\eta_{ij}^{2n+2m-1}}{(2n+2m)!} - \\
&-(\frac{\vec{\mathbf{k}}_{\mathbf{i}}}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\cdot\vec{\partial}_{ij})(\frac{\vec{\mathbf{k}}_{\mathbf{j}}}{\sqrt{m_j^2+\vec{\kappa}_j{}^2}}\cdot\vec{\partial}_{ij})
\end{aligned}
$$

$$(\frac{\vec{\kappa}_i}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\cdot\vec{\partial}_{ij})^{2m}(\frac{\vec{\kappa}_j}{\sqrt{m_j^2+\vec{\kappa}_j{}^2}}\cdot\vec{\partial}_{ij})^{2n}\frac{\eta_{ij}^{2n+2m+1}}{(2n+2m+2)!}\Big] =$$

$$= \sum_{i\neq j}^{N}\frac{Q_iQ_j}{8\pi}\Big[\frac{\vec{\mathbf{k}}_{\mathbf{i}}}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\cdot\frac{\vec{\mathbf{k}}_{\mathbf{j}}}{\sqrt{m_j^2+\vec{\kappa}_j{}^2}}(\sum_{m=0}^{\infty}\Big([(\frac{\vec{\kappa}_i}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\cdot\vec{\partial}_{ij})^{2m}+$$

$$+(\frac{\vec{\kappa}_j}{\sqrt{m_j^2+\vec{\kappa}_j{}^2}}\cdot\vec{\partial}_{ij})^{2m}]\frac{\eta_{ij}^{2m-1}}{(2m)!}\Big)-\eta_{ij}^{-1})-$$

$$-(\frac{\vec{\mathbf{k}}_{\mathbf{i}}}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\cdot\vec{\partial}_{ij})(\frac{\vec{\mathbf{k}}_{\mathbf{j}}}{\sqrt{m_j^2+\vec{\kappa}_j{}^2}}\cdot\vec{\partial}_{ij})(\sum_{m=0}^{\infty}\Big([(\frac{\vec{\kappa}_i}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\cdot\vec{\partial}_{ij})^{2m}+$$

$$+(\frac{\vec{\kappa}_j}{\sqrt{m_j^2+\vec{\kappa}_j{}^2}}\cdot\vec{\partial}_{ij})^{2m}]\frac{\eta_{ij}^{2m+1}}{(2m+2)!}\Big)-\frac{1}{2}\eta_{ij})+$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}[\frac{\vec{\mathbf{k}}_{\mathbf{i}}}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\cdot\frac{\vec{\mathbf{k}}_{\mathbf{j}}}{\sqrt{m_j^2+\vec{\kappa}_j{}^2}}$$

$$(\frac{\vec{\kappa}_i}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\cdot\vec{\partial}_{ij})^{2m+2}(\frac{\vec{\kappa}_j}{\sqrt{m_j^2+\vec{\kappa}_j{}^2}}\cdot\vec{\partial}_{ij})^{2n+2}\frac{\eta_{ij}^{2n+2m+3}}{(2n+2m+4)!}-$$

$$-(\frac{\vec{\mathbf{k}}_{\mathbf{i}}}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\cdot\vec{\partial}_{ij})(\frac{\vec{\mathbf{k}}_{\mathbf{j}}}{\sqrt{m_j^2+\vec{\kappa}_j{}^2}}\cdot\vec{\partial}_{ij})$$

$$(\frac{\vec{\kappa}_i}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\cdot\vec{\partial}_{ij})^{2m+2}(\frac{\vec{\kappa}_j}{\sqrt{m_j^2+\vec{\kappa}_j{}^2}}\cdot\vec{\partial}_{ij})^{2n+2}\frac{\eta_{ij}^{2n+2m+5}}{(2n+2m+6)!}]\Big]. \tag{D3}$$

The first three lines on the right hand side contains two separate types of summations, each of which can be summed to close forms. The first type involves

$$\sum_{m=0}^{\infty}(\frac{\vec{\kappa}_i}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\cdot\vec{\partial}_{ij})^{2m}\frac{\eta_{ij}^{2m-1}}{(2m)!}=\frac{1}{\eta_{ij}}\frac{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}{\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2}}. \tag{D4}$$

In order to perform the second type we use

$$\frac{\vec{\partial}_{ij}\vec{\partial}_{ij}}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}(\frac{\vec{\kappa}_i}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\cdot\vec{\partial}_{ij})^{2m}\frac{\eta_{ij}^{2m+1}}{(2m+2)!}$$

$$=\frac{(2m+1)!!(2m-1)!!}{\eta_{ij}(2m+2)!(\sqrt{m_i^2+\vec{\kappa}_i{}^2})^{2m+1}}\Big(\mathbf{I}[\vec{\kappa}_i^2-(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2]^m+$$

$$+2m[\vec{\kappa}_i\vec{\kappa}_i-(\vec{\kappa}_i\cdot\hat{\eta}_{ij})(\vec{\kappa}_i\hat{\eta}_{ij}+\hat{\eta}_{ij}\vec{\kappa}_i)](\vec{\kappa}_i^2-(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2)^{m-1}-$$

$$-(1-\delta_{m0})\hat{\eta}_{ij}\hat{\eta}_{ij}(\vec{\kappa}_i^2-(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2)^{m-1}(\vec{\kappa}_i^2-(2m+1)(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2)\Big). \tag{D5}$$

where $\mathbf{I}$ is the unit dyad. This complex form was obtained by examples for $m=0,1,2,3,4$. As a simple check on this result consider (for $m\neq 0$) the contraction Eq.(D5) with the dyad $\vec{\kappa}_i\vec{\kappa}_j$. This gives

$$
\begin{aligned}
&(\frac{\vec{\kappa}_i}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\cdot\vec{\partial}_{ij})^{2m+2}\frac{\eta_{ij}^{2m+1}}{(2m+2)!}=\\
&=\frac{(2m+1)!!(2m-1)!!}{\eta_{ij}(2m+2)!(\sqrt{m_i^2+\vec{\kappa}_i{}^2})^{2m+2}}\\
&\Big(\vec{\kappa}_i^2(\vec{\kappa}_i^2-(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2)^m+2m[(\vec{\kappa}_i^2)^2-2\vec{\kappa}_i^2(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2](\vec{\kappa}_i^2-(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2)^{m-1}-\\
&-(1-\delta_{m0})(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2(\vec{\kappa}_i^2-(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2)^{m-1}(\vec{\kappa}_i^2-(2m+1)(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2)\Big)=\\
&=\frac{(2m+1)!!(2m-1)!!(\vec{\kappa}_i^2-(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2)^{m-1}}{\eta_{ij}(2m+2)!(\sqrt{m_i^2+\vec{\kappa}_i{}^2})^{2m+2}}\\
&\times\Big(\vec{\kappa}_i^2(\vec{\kappa}_i^2-(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2)+2m[(\vec{\kappa}_i^2)^2-2\vec{\kappa}_i^2(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2]-\\
&-(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2(\vec{\kappa}_i^2-(2m+1)(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2)\Big)=\\
&=\frac{[(2m+1)!!]^2(\vec{\kappa}_i^2-(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2)^{m+1}}{\eta_{ij}(2m+2)!(\sqrt{m_i^2+\vec{\kappa}_i{}^2})^{2m+2}},
\end{aligned}
\tag{D6}
$$

and it agrees with the equation described above Eq.(6.32). Using Eq.(D5), the second sum (we separate out the $m=0$ term and let $x=\frac{\vec{\kappa}_i^2-(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2}{(\sqrt{m_i^2+\vec{\kappa}_i{}^2})^2}$) has three parts, the first of which is

$$
\begin{aligned}
&\sum_{m=1}^{\infty}\frac{(2m+1)!!(2m-1)!!}{\eta_{ij}(2m+2)!(\sqrt{m_i^2+\vec{\kappa}_i{}^2})^{2m+1}}\mathbf{I}[\vec{\kappa}_i^2-(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2]^m=\\
&=x\frac{\mathbf{I}}{\eta_{ij}\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\sum_{m=0}^{\infty}\frac{(-)^m x^m}{2m+3}\binom{-1/2}{m+2}=\\
&=\frac{\mathbf{I}}{\eta_{ij}\sqrt{m_i^2+\vec{\kappa}_i{}^2}}x^{1/2}\int_0^{x^{1/2}}dz\sum_{m=0}^{\infty}(-)^m z^{2m+2}\binom{-1/2}{m+2}=\\
&=\frac{\mathbf{I}}{\eta_{ij}\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\frac{1-x/2-\sqrt{1-x}}{x}\\
&=\frac{\mathbf{I}}{\eta_{ij}\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\Big[-\frac{1}{2}-\frac{\sqrt{m_i^2+\vec{\kappa}_i{}^2}\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2}}{\vec{\kappa}_i^2-(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2}+\frac{(\sqrt{m_i^2+\vec{\kappa}_i{}^2})^2}{\vec{\kappa}_i^2-(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2}\Big]=\\
&=\frac{\mathbf{I}}{2\eta_{ij}\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\frac{\sqrt{m_i^2+\vec{\kappa}_i{}^2}-\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2}}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}+\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2}},
\end{aligned}
\tag{D7}
$$

while the second part is

$$
\begin{aligned}
&\sum_{m=1}^{\infty}\frac{(2m+1)!!(2m-1)!!}{\eta_{ij}(2m+2)!(\sqrt{m_i^2+\vec{\kappa}_i{}^2})^{2m+1}}\\
&2m\Big[\vec{\kappa}_i\vec{\kappa}_i-(\vec{\kappa}_i\cdot\hat{\eta}_{ij})(\vec{\kappa}_i\hat{\eta}_{ij}+\hat{\eta}_{ij}\vec{\kappa}_i)\Big](\vec{\kappa}_i^2-(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2)^{m-1}=\\
&=\frac{[\vec{\kappa}_i\vec{\kappa}_i-(\vec{\kappa}_i\cdot\hat{\eta}_{ij})(\vec{\kappa}_i\hat{\eta}_{ij}+\hat{\eta}_{ij}\vec{\kappa}_i)]}{\eta_{ij}(\vec{\kappa}_i^2-(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2)\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\sum_{m=1}^{\infty}\frac{(2m+1)!!(2m-1)!!}{(2m+2)!}2mx^m.
\end{aligned}
\tag{D8}
$$

Using

$$\sum_{m=1}^{\infty}\frac{(2m+1)!!(2m-1)!!}{(2m+2)!}2mx^m =$$
$$= x\sum_{m=0}^{\infty}(-)^m x^m\binom{-1/2}{m+2}\frac{2m+2}{2m+3} =$$
$$= x\sum_{m=0}^{\infty}(-)^m x^m\binom{-1/2}{m+2} - x\sum_{m=0}^{\infty}(-)^m x^m\binom{-1/2}{m+2}\frac{1}{2m+3} =$$
$$= \frac{1}{x}[\frac{1}{\sqrt{1-x}} - 1 - \frac{x}{2} - 1 + x/2 + \sqrt{1-x}] =$$
$$= \frac{1}{x}[\frac{1}{\sqrt{1-x}} - 2 + \sqrt{1-x}]. \tag{D9}$$

Eq.(D8) becomes

$$= \frac{[\vec{\kappa}_i\vec{\kappa}_i - (\vec{\kappa}_i\cdot\hat{\eta}_{ij})(\vec{\kappa}_i\hat{\eta}_{ij} + \hat{\eta}_{ij}\vec{\kappa}_i)]\sqrt{m_i^2+\vec{\kappa}_i{}^2}}{\eta_{ij}(\vec{\kappa}_i^2 - (\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2)^2}[\frac{\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2}}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}} + \frac{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}{\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2}} - 2] =$$
$$= \frac{[\vec{\kappa}_i\vec{\kappa}_i - (\vec{\kappa}_i\cdot\hat{\eta}_{ij})(\vec{\kappa}_i\hat{\eta}_{ij} + \hat{\eta}_{ij}\vec{\kappa}_i)]}{\eta_{ij}\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2}(\sqrt{m_i^2+\vec{\kappa}_i{}^2} + \sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2})^2}. \tag{D10}$$

Finally our third part of the sum is

$$-\frac{\hat{\eta}_{ij}\hat{\eta}_{ij}}{\eta_{ij}}\sum_{m=1}^{\infty}\frac{(2m+1)!!(2m-1)!!}{(2m+2)!\Big(\sqrt{m_i^2+\vec{\kappa}_i{}^2}\Big)^{2m+1}}\Big(\vec{\kappa}_i^2 - (\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2\Big)^{m-1}\Big(\vec{\kappa}_i^2 - (2m+1)(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2\Big) =$$
$$= -\frac{\hat{\eta}_{ij}\hat{\eta}_{ij}}{\eta_{ij}}\frac{1}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}(\vec{\kappa}_i^2 - (\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2)}x$$
$$\Big[\vec{\kappa}_i^2\sum_{m=0}^{\infty}\frac{(-)^m x^m}{2m+3}\binom{-1/2}{m+2} - (\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2\sum_{m=0}^{\infty}(-)^m x^m\binom{-1/2}{m+2}\Big]. \tag{D11}$$

Using the results above we get

$$x\Big[\vec{\kappa}_i^2\sum_{m=0}^{\infty}\frac{(-)^m x^m}{2m+3}\binom{-1/2}{m+2} - (\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2\sum_{m=0}^{\infty}(-)^m x^m\binom{-1/2}{m+2}\Big] =$$
$$= -\frac{(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2}{x}[\frac{1}{\sqrt{1-x}} - 1 - \frac{x}{2}] + \frac{\vec{\kappa}_i^2}{x}[1 - x/2 - \sqrt{1-x}] =$$
$$= \frac{\sqrt{m_i^2+\vec{\kappa}_i{}^2} - \sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2}}{\sqrt{m_i^2+\vec{\kappa}_i{}^2} + \sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2}}\Big[\frac{\vec{\kappa}_i^2}{2} - \frac{(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2(2\sqrt{m_i^2+\vec{\kappa}_i{}^2} + \sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2})}{2\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2}}\Big], \tag{D12}$$

so that our third part becomes

$$-\frac{\hat{\eta}_{ij}\hat{\eta}_{ij}}{\eta_{ij}}\sum_{m=1}^{\infty}\frac{(2m+1)!!(2m-1)!!}{(2m+2)!(\sqrt{m_i^2+\vec{\kappa}_i{}^2})^{2m+1}}(\vec{\kappa}_i^2 - (\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2)^{m-1}(\vec{\kappa}_i^2 - (2m+1)(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2) =$$

$$= -\frac{\hat{\eta}_{ij}\hat{\eta}_{ij}}{\eta_{ij}\sqrt{m_i^2+\vec{\kappa_i}^2}} \frac{1}{(\sqrt{m_i^2+\vec{\kappa_i}^2}+\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2})^2}$$

$$\Big[\frac{\vec{\kappa}_i^2}{2} - \frac{(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2(2\sqrt{m_i^2+\vec{\kappa_i}^2}+\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2})}{2\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2}}\Big] -$$

$$= -\frac{\hat{\eta}_{ij}\hat{\eta}_{ij}}{\eta_{ij}\sqrt{m_i^2+\vec{\kappa_i}^2}} \Big[\frac{\sqrt{m_i^2+\vec{\kappa_i}^2}-\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2}}{2(\sqrt{m_i^2+\vec{\kappa_i}^2}+\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2})} -$$

$$-\frac{(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2\sqrt{m_i^2+\vec{\kappa_i}^2}}{(\sqrt{m_i^2+\vec{\kappa_i}^2}+\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2})^2\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2}}\Big]. \tag{D13}$$

Combining these three parts of $U(\tau)$ gives us

$$\sum_{i\neq j}^{N}\frac{Q_iQ_j}{8\pi}\Big[\frac{\vec{\mathbf{k}}_\mathbf{i}}{\sqrt{m_i^2+\vec{\kappa_i}^2}}\cdot\frac{\vec{\mathbf{k}}_\mathbf{j}}{\sqrt{m_j^2+\vec{\kappa_j}^2}}$$

$$(\sum_{m=0}^{\infty}\Big([(\frac{\vec{\kappa}_i}{\sqrt{m_i^2+\vec{\kappa_i}^2}}\cdot\vec{\partial}_{ij})^{2m}+(\frac{\vec{\kappa}_j}{\sqrt{m_j^2+\vec{\kappa_j}^2}}\cdot\vec{\partial}_{ij})^{2m}]\frac{\eta_{ij}^{2m-1}}{(2m)!}-\eta_{ij}^{-1}\Big)) -$$

$$-(\frac{\vec{\mathbf{k}}_\mathbf{i}}{\sqrt{m_i^2+\vec{\kappa_i}^2}}\cdot\vec{\partial}_{ij})(\frac{\vec{\mathbf{k}}_\mathbf{j}}{\sqrt{m_j^2+\vec{\kappa_j}^2}}\cdot\vec{\partial}_{ij})(\sum_{m=0}^{\infty}\Big([(\frac{\vec{\kappa}_i}{\sqrt{m_i^2+\vec{\kappa_i}^2}}\cdot\vec{\partial}_{ij})^{2m}+$$

$$+(\frac{\vec{\kappa}_j}{\sqrt{m_j^2+\vec{\kappa_j}^2}}\cdot\vec{\partial}_{ij})^{2m}]\frac{\eta_{ij}^{2m+1}}{(2m+2)!}-\frac{1}{2}\eta_{ij}\Big)\Big] =$$

$$= \sum_{i\neq j}^{N}\frac{Q_iQ_j}{8\pi}\Big[\frac{\vec{\mathbf{k}}_\mathbf{i}}{\sqrt{m_i^2+\vec{\kappa_i}^2}}\cdot\frac{\vec{\mathbf{k}}_\mathbf{j}}{\sqrt{m_j^2+\vec{\kappa_j}^2}}\frac{1}{\eta_{ij}}\Big(\frac{\sqrt{m_i^2+\vec{\kappa_i}^2}}{\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2}}+\frac{\sqrt{m_j^2+\vec{\kappa_j}^2}}{\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2}}-1\Big) -$$

$$-\frac{\vec{\mathbf{k}}_\mathbf{i}}{2\sqrt{m_i^2+\vec{\kappa_i}^2}}\cdot\frac{\vec{\mathbf{k}}_\mathbf{j}}{\sqrt{m_j^2+\vec{\kappa_j}^2}}\frac{1}{\eta_{ij}}+\frac{\vec{\mathbf{k}}_\mathbf{i}}{2\sqrt{m_i^2+\vec{\kappa_i}^2}}\cdot(\frac{\vec{\mathbf{k}}_\mathbf{j}}{\sqrt{m_j^2+\vec{\kappa_j}^2}}\cdot\frac{\hat{\eta}_{ij}}{\eta_{ij}}\hat{\eta}_{ij}) -$$

$$-\frac{\vec{\mathbf{k}}_\mathbf{i}\cdot\vec{\mathbf{k}}_\mathbf{j}}{\sqrt{m_i^2+\vec{\kappa_i}^2}\sqrt{m_j^2+\vec{\kappa_j}^2}}\Big(\frac{1}{2\eta_{ij}}\frac{\sqrt{m_i^2+\vec{\kappa_i}^2}-\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2}}{\sqrt{m_i^2+\vec{\kappa_i}^2}+\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2}}+$$

$$+\frac{1}{2\eta_{ij}(\sqrt{m_j^2+\vec{\kappa_j}^2})^2}\frac{\sqrt{m_j^2+\vec{\kappa_j}^2}-\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2}}{\sqrt{m_j^2+\vec{\kappa_j}^2}+\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2}}\Big) -$$

$$-\frac{\vec{\mathbf{k}}_\mathbf{i}\cdot\Big(\vec{\mathbf{k}}_\mathbf{j}\cdot}{\sqrt{m_j^2+\vec{\kappa_j}^2}}\frac{[\vec{\kappa}_i\vec{\kappa}_i-(\vec{\kappa}_i\cdot\hat{\eta}_{ij})(\vec{\kappa}_i\hat{\eta}_{ij}+\hat{\eta}_{ij}\vec{\kappa}_i)]}{\eta_{ij}\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2}}\frac{1}{(\sqrt{m_i^2+\vec{\kappa_i}^2}+\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2})^2}\Big) -$$

$$-\frac{\vec{\mathbf{k}}_\mathbf{i}\cdot\Big(\vec{\mathbf{k}}_\mathbf{j}\cdot}{\sqrt{m_i^2+\vec{\kappa_i}^2}}\frac{[\vec{\kappa}_j\vec{\kappa}_j-(\vec{\kappa}_j\cdot\hat{\eta}_{ij})(\vec{\kappa}_j\hat{\eta}_{ij}+\hat{\eta}_{ij}\vec{\kappa}_j)]}{\eta_{ij}\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2}}\frac{1}{(\sqrt{m_j^2+\vec{\kappa_j}^2}+\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2})^2}\Big) +$$

$$+\frac{\vec{\mathbf{k}}_\mathbf{i}\cdot\Big(\vec{\mathbf{k}}_\mathbf{j}\cdot}{\sqrt{m_i^2+\vec{\kappa_i}^2}\sqrt{m_j^2+\vec{\kappa_j}^2}}\frac{\hat{\eta}_{ij}\hat{\eta}_{ij}}{\eta_{ij}}$$

$$
\Big(\frac{\sqrt{m_i^2+\vec{\kappa}_i{}^2}-\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2}}{2(\sqrt{m_i^2+\vec{\kappa}_i{}^2}+\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2})}-
$$

$$
-\frac{(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2\sqrt{m_i^2+\vec{\kappa}_i{}^2}}{(\sqrt{m_i^2+\vec{\kappa}_i{}^2}+\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2})^2\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2}}\Big)\Big)+
$$

$$
+\frac{\vec{\mathbf{k}}_{\mathbf{i}}\cdot\big(\vec{\mathbf{k}}_{\mathbf{j}}\cdot}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}\sqrt{m_j^2+\vec{\kappa}_j{}^2}}\frac{\hat{\eta}_{ij}\hat{\eta}_{ij}}{\eta_{ij}}
$$

$$
\Big(\frac{\sqrt{m_j^2+\vec{\kappa}_j{}^2}-\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2}}{2(\sqrt{m_j^2+\vec{\kappa}_j{}^2}+\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2})}-
$$

$$
-\frac{(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2\sqrt{m_j^2+\vec{\kappa}_j{}^2}}{(\sqrt{m_j^2+\vec{\kappa}_j{}^2}+\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2})^2\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2}}\Big)\Big)\Big]. \tag{D14}
$$

Rearrangement leads to

$$
\sum_{i\neq j}^{N}\frac{Q_iQ_j}{8\pi}\Big[\frac{\vec{\mathbf{k}}_{\mathbf{i}}\cdot\vec{\mathbf{k}}_{\mathbf{j}}}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}\sqrt{m_j^2+\vec{\kappa}_j{}^2}}\frac{1}{\eta_{ij}}\Big(\frac{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}{\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2}}+\frac{\sqrt{m_j^2+\vec{\kappa}_j{}^2}}{\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2}}-1\Big)+
$$

$$
+\frac{\vec{\mathbf{k}}_{\mathbf{i}}\cdot\vec{\mathbf{k}}_{\mathbf{j}}-\vec{\mathbf{k}}_{\mathbf{i}}\cdot(\vec{\mathbf{k}}_{\mathbf{j}}\cdot(\hat{\eta}_{\mathbf{ij}}\hat{\eta}_{\mathbf{ij}}))}{2\sqrt{m_i^2+\vec{\kappa}_i{}^2}\sqrt{m_j^2+\vec{\kappa}_j{}^2}}\frac{1}{\eta_{ij}}-
$$

$$
-\frac{(\vec{\mathbf{k}}_i\cdot\vec{\mathbf{k}}_j-\vec{\mathbf{k}}_i\cdot(\vec{\mathbf{k}}_j\cdot(\hat{\eta}_{ij}\hat{\eta}_{ij})))}{\sqrt{m_j^2+\vec{\kappa}_j{}^2}}\frac{1}{\eta_{ij}(\sqrt{m_i^2+\vec{\kappa}_i{}^2}+\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2})}-
$$

$$
-\frac{(\vec{\mathbf{k}}_i\cdot\vec{\mathbf{k}}_j-\vec{\mathbf{k}}_i\cdot(\vec{\mathbf{k}}_j\cdot(\hat{\eta}_{ij}\hat{\eta}_{ij})))}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\frac{1}{\eta_{ij}(\sqrt{m_j^2+\vec{\kappa}_j{}^2}+\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2})}-
$$

$$
-\Big((\vec{\mathbf{k}}_i\cdot\vec{\kappa}_i)(\vec{\mathbf{k}}_j\cdot\vec{\kappa}_i-(\vec{\mathbf{k}}_j\cdot\hat{\eta}_{ij})(\vec{\kappa}_i\cdot\hat{\eta}_{ij}))-\vec{\mathbf{k}}_j\cdot[(\hat{\vec{k}}_i\cdot\hat{\eta}_{ij})(\vec{\kappa}_i-\hat{\eta}_{ij}(\vec{\kappa}_i\cdot\hat{\eta}_{ij}))](\vec{\kappa}_i\cdot\hat{\eta}_{ij})\Big)
$$

$$
\frac{1}{\eta_{ij}\sqrt{m_j^2+\vec{\kappa}_j{}^2}\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2}}\frac{1}{(\sqrt{m_i^2+\vec{\kappa}_i{}^2}+\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2})^2}-
$$

$$
-\Big((\vec{\mathbf{k}}_j\cdot\vec{\kappa}_j)(\vec{\mathbf{k}}_i\cdot\vec{\kappa}_j-(\vec{\mathbf{k}}_i\cdot\hat{\eta}_{ij})(\vec{\kappa}_j\cdot\hat{\eta}_{ij}))-\vec{\mathbf{k}}_i\cdot[(\hat{\vec{k}}_j\cdot\hat{\eta}_{ij})(\vec{\kappa}_j-\hat{\eta}_{ij}(\vec{\kappa}_j\cdot\hat{\eta}_{ij}))](\vec{\kappa}_j\cdot\hat{\eta}_{ij})\Big)
$$

$$
\frac{1}{\eta_{ij}\sqrt{m_i^2+\vec{\kappa}_i{}^2}\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2}}\frac{1}{(\sqrt{m_j^2+\vec{\kappa}_j{}^2}+\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2})^2}\Big]. \tag{D15}
$$

Note that the double dyadic dot products are defined such that the right most $k's$ are contracted with the right most $\hat{\eta}'s$.

Now we combine this with the expression (7.7) for the corresponding portion of U_1 (involving $\vec{A}_\perp$). First note that

$$
-\sum_{i=1}^{N}\frac{Q_i\vec{\kappa}_i(\tau)\cdot\vec{A}_\perp(\tau,\vec{\eta}_i(\tau))}{\sqrt{m_i^2+\check{\vec{\kappa}}_i(\tau)^2}}=
$$

$$= -\sum_{i\neq j}^{N} \frac{Q_j Q_i}{4\pi\sqrt{m_i^2+\vec{\kappa}_i{}^2}} \Big[\Big(\vec{\kappa}_i + \frac{i\vec{\kappa}_j\cdot\vec{\xi}_j\vec{\kappa}_i\cdot\vec{\xi}_j\vec{\partial}_{ij}}{\sqrt{m_j^2+\vec{\kappa}_j^2}(m_j+\sqrt{m_j^2+\vec{\kappa}_j^2})}\Big)\cdot[\vec{\kappa}_j+\hat{\eta}_{ij}\vec{\kappa}_j\cdot\hat{\eta}_{ij}]$$

$$\frac{1}{|\vec{\eta}_i-\vec{\eta}_j|(\sqrt{m_j^2+\vec{\kappa}_j^2}+\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2})} - \vec{\kappa}_i\cdot\vec{\xi}_j\vec{\xi}_j\cdot\vec{\partial}_{ij}\frac{1}{|\vec{\eta}_i-\vec{\eta}_j|\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2}}\Big] =$$

$$= -\sum_{i\neq j}^{N}\frac{Q_jQ_i}{4\pi}\Big[\vec{\kappa}_i\cdot\vec{k}_j\frac{1}{\eta_{ij}\sqrt{m_i^2+\vec{\kappa}_i{}^2}\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2}} -$$

$$-\vec{\kappa}_i\cdot\Big(\vec{\mathbf{l}}_j\cdot(\mathbf{I}-\hat{\eta}_{ij}\hat{\eta}_{ij})\frac{1}{\eta_{ij}\sqrt{m_i^2+\vec{\kappa}_i{}^2}\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2}}(\frac{\sqrt{m_j^2+\vec{\kappa}_j^2}}{\sqrt{m_j^2+\vec{\kappa}_j^2}+\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2}})\Big)\Big], \tag{D16}$$

or

$$= -\sum_{i\neq j}^{N}\frac{Q_jQ_i}{8\pi}\Big[\frac{\vec{\kappa}_i\cdot\vec{\mathbf{k}}_j}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\frac{1}{\eta_{ij}\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2}} +$$

$$+\frac{\vec{\kappa}_j\cdot\vec{\mathbf{k}}_i}{\sqrt{m_j^2+\vec{\kappa}_j^2}}\frac{1}{\eta_{ij}\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2}} -$$

$$-[\vec{\kappa}_i\cdot\vec{\mathbf{l}}_j-\vec{\kappa}_i\cdot(\hat{\vec{l}}_j\cdot(\hat{\eta}_{ij}\hat{\eta}_{ij}))]$$

$$\frac{1}{\eta_{ij}\sqrt{m_i^2+\vec{\kappa}_i{}^2}\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2}}(\frac{\sqrt{m_j^2+\vec{\kappa}_j^2}}{\sqrt{m_j^2+\vec{\kappa}_j^2}+\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2}}) -$$

$$-[\vec{\kappa}_j\cdot\vec{\mathbf{l}}_i-\vec{\kappa}_j\cdot(\hat{\vec{l}}_i\cdot(\hat{\eta}_{ij}\hat{\eta}_{ij}))]$$

$$\frac{1}{\eta_{ij}\sqrt{m_j^2+\vec{\kappa}_j^2}\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2}}(\frac{\sqrt{m_i^2+\vec{\kappa}_i^2}}{\sqrt{m_i^2+\vec{\kappa}_i^2}+\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2}})\Big]. \tag{D17}$$

Now combine Eq.(D17) with the above term Eq.(D15) from $U(\tau)$. In order to clarify the cancellations that will take place consider the case without spin for which the spin operator containing terms $\vec{\mathbf{k}}_i$ and $\vec{\mathbf{l}}_i$ reduce to the simple momentum factor $\vec{\kappa}_i$. In that case Eq.(D15) becomes

$$= \sum_{i\neq j}^{N}\frac{Q_iQ_j}{8\pi}\Big[\frac{\vec{\kappa}_i\cdot\vec{\kappa}_j}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}\sqrt{m_j^2+\vec{\kappa}_j{}^2}}\frac{1}{\eta_{ij}}\Big(\frac{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}{\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2}}+\frac{\sqrt{m_j^2+\vec{\kappa}_j{}^2}}{\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2}}-1\Big) +$$

$$+\frac{\vec{\kappa}_i\cdot\vec{\kappa}_j-\vec{\kappa}_i\cdot\hat{\eta}_{ij}\vec{\kappa}_j\cdot\hat{\eta}_{ij}}{\eta_{ij}}\Big(\frac{1}{2\sqrt{m_i^2+\vec{\kappa}_i{}^2}\sqrt{m_j^2+\vec{\kappa}_j{}^2}} -$$

$$-\frac{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}{\sqrt{m_j^2+\vec{\kappa}_j{}^2}\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2}(\sqrt{m_i^2+\vec{\kappa}_i{}^2}+\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2})} -$$

$$-\frac{\sqrt{m_j^2+\vec{\kappa}_j{}^2}}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2}(\sqrt{m_j^2+\vec{\kappa}_j{}^2}+\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2})}\Big)\Big], \tag{D18}$$

while the corresponding portion of $U_1(\tau)$ becomes

$$= -\sum_{i\neq j}^{N} \frac{Q_j Q_i}{8\pi}\Big[\frac{\vec{\kappa}_i\cdot\vec{\kappa}_j}{\sqrt{m_j^2+\vec{\kappa}_j^2}}\frac{1}{\eta_{ij}\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2}} +$$
$$+\frac{\vec{\kappa}_j\cdot\vec{\kappa}_j}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\frac{1}{\eta_{ij}\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2}} -$$
$$-[\vec{\kappa}_i\cdot\vec{\kappa}_j-\vec{\kappa}_i\cdot\hat{\eta}_{ij}\vec{\kappa}_j\cdot\hat{\eta}_{ij}]\frac{1}{\eta_{ij}\sqrt{m_i^2+\vec{\kappa}_i^2}\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2}}$$
$$(\frac{\sqrt{m_j^2+\vec{\kappa}_j^2}}{\sqrt{m_j^2+\vec{\kappa}_j^2}+\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2}}) -$$
$$-[\vec{\kappa}_i\cdot\vec{\kappa}_j-\vec{\kappa}_i\cdot\hat{\eta}_{ij}\vec{\kappa}_j\cdot\hat{\eta}_{ij}]\frac{1}{\eta_{ij}\sqrt{m_j^2+\vec{\kappa}_j^2}\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2}}$$
$$(\frac{\sqrt{m_i^2+\vec{\kappa}_i^2}}{\sqrt{m_i^2+\vec{\kappa}_i^2}+\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2}})\Big]. \tag{D19}$$

Adding the two gives

$$V_{LDO}^{(spinless)}(\tau) = -\sum_{i\neq j}^{N}\frac{Q_iQ_j}{16\pi}\frac{\vec{\kappa}_i\cdot\vec{\kappa}_j+\vec{\kappa}_i\cdot\hat{\eta}_{ij}\vec{\kappa}_j\cdot\hat{\eta}_{ij}}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}\sqrt{m_j^2+\vec{\kappa}_j{}^2}}\ \frac{1}{\eta_{ij}}, \tag{D20}$$

which is our relativistic extension of the standard Darwin interaction for spinless particles [5]

Now that we have checked our results without spin, we combine the spin-dependent terms of Eqs.(D15), (D17) to get ($\vec{\mathbf{k}} := \vec{\kappa}+\vec{\mathbf{f}}; \vec{\mathbf{l}} := \vec{\kappa}+\vec{\mathbf{g}}$)

$$\sum_{i\neq j}^{N}\frac{Q_iQ_j}{8\pi}\Big[-\frac{\vec{\kappa}_i\cdot\vec{\kappa}_j+\vec{\kappa}_i\cdot\hat{\eta}_{ij}\vec{\kappa}_j\cdot\hat{\eta}_{ij}}{2\sqrt{m_i^2+\vec{\kappa}_i{}^2}\sqrt{m_j^2+\vec{\kappa}_j{}^2}}\frac{1}{\eta_{ij}} +$$
$$+\Big(\frac{\vec{\mathbf{f}}_{\mathbf{i}}\cdot(\vec{\kappa}_{\mathbf{j}}+\vec{\mathbf{f}}_{\mathbf{j}})}{\sqrt{m_j^2+\vec{\kappa}_j{}^2}\eta_{ij}\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2}}+\frac{\vec{\mathbf{f}}_{\mathbf{j}}\cdot(\vec{\kappa}_{\mathbf{i}}+\vec{\mathbf{f}}_{\mathbf{i}})}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}\eta_{ij}\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2}} -$$
$$-\frac{\vec{\kappa}_i\cdot\vec{\mathbf{f}}_j+\vec{\kappa}_j\cdot\hat{\vec{f}}_i+\vec{\mathbf{f}}_i\cdot\vec{\mathbf{f}}_j}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}\sqrt{m_j^2+\vec{\kappa}_j{}^2}}\frac{1}{\eta_{ij}}\Big) +$$
$$+\Big(\vec{\kappa}_i\cdot\vec{\mathbf{f}}_j+\vec{\kappa}_j\cdot\hat{\vec{f}}_i+\vec{\mathbf{f}}_i\cdot\vec{\mathbf{f}}_j-(\vec{\mathbf{f}}_j\vec{\kappa}_i\cdot\hat{\eta}_{ij}+\vec{\mathbf{f}}_i\vec{\kappa}_j\cdot\hat{\eta}_{ij}+\vec{\mathbf{f}}_j\vec{\mathbf{f}}_i\cdot\hat{\eta}_{ij})\cdot\hat{\eta}_{ij}\Big)$$
$$\Big(\frac{1}{2\sqrt{m_i^2+\vec{\kappa}_i{}^2}\sqrt{m_j^2+\vec{\kappa}_j{}^2}\eta_{ij}} -$$
$$-\frac{1}{\eta_{ij}\sqrt{m_j^2+\vec{\kappa}_j{}^2}(\sqrt{m_i^2+\vec{\kappa}_i{}^2}+\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2})} -$$

$$-\frac{1}{\sqrt{m_i^2+\vec{\kappa}_i{}^2}\eta_{ij}(\sqrt{m_j^2+\vec{\kappa}_j{}^2}+\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2})}) -$$

$$-\Big((\vec{\kappa}_i\cdot\vec{\kappa}_i)(\vec{\mathbf{f}}_j\cdot\vec{\kappa}_i-(\vec{\mathbf{f}}_j\cdot\hat{\eta}_{ij})(\vec{\kappa}_i\cdot\hat{\eta}_{ij}))-\vec{\mathbf{f}}_j\cdot[(\vec{\kappa}_i\cdot\hat{\eta}_{ij})(\vec{\kappa}_i-\hat{\eta}_{ij}(\vec{\kappa}_i\cdot\hat{\eta}_{ij}))](\vec{\kappa}_i\cdot\hat{\eta}_{ij})+$$

$$+(\vec{\mathbf{f}}_i\cdot\vec{\kappa}_i)(\vec{\kappa}_j\cdot\vec{\kappa}_i-(\vec{\kappa}_j\cdot\hat{\eta}_{ij})(\vec{\kappa}_i\cdot\hat{\eta}_{ij}))-\vec{\kappa}_j\cdot[(\vec{\mathbf{f}}_i\cdot\hat{\eta}_{ij})(\vec{\kappa}_i-\hat{\eta}_{ij}(\vec{\kappa}_i\cdot\hat{\eta}_{ij}))](\vec{\kappa}_i\cdot\hat{\eta}_{ij})+$$

$$+(\vec{\mathbf{f}}_i\cdot\vec{\kappa}_i)(\vec{\mathbf{f}}_j\cdot\vec{\kappa}_i-(\vec{\mathbf{f}}_j\cdot\hat{\eta}_{ij})(\vec{\kappa}_i\cdot\hat{\eta}_{ij}))-\vec{\mathbf{f}}_j\cdot[(\vec{\mathbf{f}}_i\cdot\hat{\eta}_{ij})(\vec{\kappa}_i-\hat{\eta}_{ij}(\vec{\kappa}_i\cdot\hat{\eta}_{ij}))](\vec{\kappa}_i\cdot\hat{\eta}_{ij})\Big)$$

$$\frac{1}{\eta_{ij}\sqrt{m_j^2+\vec{\kappa}_j{}^2}\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2}}\frac{1}{(\sqrt{m_i^2+\vec{\kappa}_i{}^2}+\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2})^2}-$$

$$-\Big((\vec{\kappa}_j\cdot\vec{\kappa}_j)(\vec{\mathbf{f}}_i\cdot\vec{\kappa}_j-(\vec{\mathbf{f}}_i\cdot\hat{\eta}_{ij})(\vec{\kappa}_j\cdot\hat{\eta}_{ij}))-\vec{\mathbf{f}}_i\cdot[(\vec{\kappa}_j\cdot\hat{\eta}_{ij})(\vec{\kappa}_j-\hat{\eta}_{ij}(\vec{\kappa}_j\cdot\hat{\eta}_{ij}))](\vec{\kappa}_j\cdot\hat{\eta}_{ij})+$$

$$+(\vec{\mathbf{f}}_j\cdot\vec{\kappa}_j)(\vec{\kappa}_i\cdot\vec{\kappa}_j-(\vec{\kappa}_i\cdot\hat{\eta}_{ij})(\vec{\kappa}_j\cdot\hat{\eta}_{ij}))-\vec{\kappa}_i\cdot[(\vec{\mathbf{f}}_j\cdot\hat{\eta}_{ij})(\vec{\kappa}_j-\hat{\eta}_{ij}(\vec{\kappa}_j\cdot\hat{\eta}_{ij}))](\vec{\kappa}_j\cdot\hat{\eta}_{ij})+$$

$$+(\vec{\mathbf{f}}_j\cdot\vec{\kappa}_j)(\vec{\mathbf{f}}_i\cdot\vec{\kappa}_j-(\vec{\mathbf{f}}_i\cdot\hat{\eta}_{ij})(\vec{\kappa}_j\cdot\hat{\eta}_{ij}))-\vec{\mathbf{f}}_i\cdot[(\vec{\mathbf{f}}_j\cdot\hat{\eta}_{ij})(\vec{\kappa}_j-\hat{\eta}_{ij}(\vec{\kappa}_j\cdot\hat{\eta}_{ij}))](\vec{\kappa}_j\cdot\hat{\eta}_{ij})\Big)$$

$$\frac{1}{\eta_{ij}\sqrt{m_i^2+\vec{\kappa}_i{}^2}\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2}}\frac{1}{(\sqrt{m_j^2+\vec{\kappa}_j{}^2}+\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2})^2}+$$

$$+\Big(\vec{\kappa}_i\cdot\vec{\mathbf{g}}_j-\vec{\kappa}_i\cdot(\hat{\vec{g}}_j\cdot(\hat{\eta}_{ij}\hat{\eta}_{ij}))\Big)$$

$$\frac{1}{\eta_{ij}\sqrt{m_i^2+\vec{\kappa}_i^2}\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2}}(\frac{\sqrt{m_j^2+\vec{\kappa}_j^2}}{\sqrt{m_j^2+\vec{\kappa}_j^2}+\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2}})+$$

$$+\Big(\vec{\kappa}_j\cdot\vec{\mathbf{g}}_i-\vec{\kappa}_j\cdot(\hat{\vec{g}}_i\cdot(\hat{\eta}_{ij}\hat{\eta}_{ij}))\Big)$$

$$\frac{1}{\eta_{ij}\sqrt{m_j^2+\vec{\kappa}_j^2}\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2}}(\frac{\sqrt{m_i^2+\vec{\kappa}_i^2}}{\sqrt{m_i^2+\vec{\kappa}_i^2}+\sqrt{m_i^2+(\vec{\kappa}_i\cdot\hat{\eta}_{ij})^2}})\Big], \tag{D21}$$

in which

$$\vec{\mathbf{f}}_i=-i\vec{\xi}_i\vec{\xi}_i\cdot\vec{\partial}_{ij}+\frac{i\vec{\kappa}_i\cdot\vec{\partial}_{ij}\vec{\kappa}_i\cdot\vec{\xi}_i\vec{\xi}_i}{(m_i+\sqrt{m_i^2+\vec{\kappa}_i{}^2})\sqrt{m_i^2+\vec{\kappa}_i{}^2}},$$

$$\vec{\mathbf{g}}_i=\frac{i\vec{\kappa}_i\cdot\vec{\partial}_{ij}\vec{\kappa}_i\cdot\vec{\xi}_i\vec{\xi}_i}{(m_i+\sqrt{m_i^2+\vec{\kappa}_i{}^2})\sqrt{m_i^2+\vec{\kappa}_i{}^2}}. \tag{D22}$$

To this we must add the remaining parts of $U_1(\tau)$ and $U(\tau)$. To complete $U_1(\tau)$ we need the expression for $\mathcal{R}_i$ in Eq.(7.4). The magnetic portion (7.8) is

$$\sum_{i=1}^N\frac{iQ_i\vec{\xi}_i\times\vec{\xi}_i\cdot\check{\vec{B}}(\tau,\vec{\eta}_i)}{2\sqrt{m_i^2+\check{\vec{\kappa}}_i^2}}=-\sum_{i\neq j}^N\frac{iQ_iQ_j\vec{\xi}_i\times\vec{\xi}_i\cdot(\vec{\mathbf{k}}_j\times\vec{\partial}_{ij})}{8\pi\sqrt{m_i^2+\vec{\kappa}_i{}^2}}\Big(\frac{1}{|\vec{\eta}_i-\vec{\eta}_j|}\frac{1}{\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2}}\Big), \tag{D23}$$

while the transverse electric part (7.9) is

$$-i\sum_{i=1}^N\frac{Q_i\check{\vec{\kappa}}_i\cdot\vec{\xi}_i\vec{\xi}_i\cdot\check{\vec{\pi}}_\perp(\tau,\vec{\eta}_i)}{\sqrt{m_i^2+\check{\vec{\kappa}}_i{}^2}(\sqrt{m_i^2+\check{\vec{\kappa}}_i^2}+m_i)}=$$

$$=i\sum_{i\neq j}^N\frac{Q_iQ_j\vec{\kappa}_i\cdot\vec{\xi}_i}{4\pi\sqrt{m_i^2+\vec{\kappa}_i{}^2}(\sqrt{m_i^2+\vec{\kappa}_i^2}+m_i)}\Big[\Big(\vec{\xi}_i+\frac{i\vec{\kappa}_j\cdot\vec{\xi}_j\vec{\xi}_i\cdot\vec{\xi}_j\vec{\partial}_{ij}}{\sqrt{m_j^2+\vec{\kappa}_j^2}(\sqrt{m_j^2+\vec{\kappa}_j^2}+m_j)}\Big)\cdot\frac{\hat{\eta}_{ij}}{\eta_{ij}^2}$$

$$\times(\frac{m_j^2\sqrt{m_j^2+\vec{\kappa}_j^2}}{[m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2]^{3/2}}-1)-i\vec{\xi}_i\cdot\vec{\xi}_j\vec{\xi}_j\cdot\vec{\partial}_{ij}\vec{\kappa}_j\cdot\vec{\partial}_{ij}\frac{1}{\eta_{ij}}\frac{1}{\sqrt{m_j^2+(\vec{\kappa}_j\cdot\hat{\eta}_{ij})^2}}\Big]. \tag{D24}$$

Combining Eq.(D23) and Eq.(D24) with Eq.(D21), with Eq.(7.5) for V_{DSO} and with the first four lines of Eq.(D1) produces Eq.(7.11).

APPENDIX E: SUMMATION OF TWO-BODY REST ENERGY TO CLOSED FORM

In the special case of the two body system we can obtain a closed form if we use the rest frame condition $\tilde{\vec{\kappa}}_1 + \tilde{\vec{\kappa}}_2 = 0$. The expression we get in this way may be used with the Dirac brackets associated with $\tilde{\vec{\kappa}}_1 + \tilde{\vec{\kappa}}_2 \approx 0$, so that the final reduced phase contains only $\tilde{\eta} = |\tilde{\vec{\eta}}_1 - \tilde{\vec{\eta}}_2|$ and $\tilde{\vec{\kappa}} := \tilde{\vec{\kappa}}_1 = -\tilde{\vec{\kappa}}_2$. Using the identity below Eq.(6.32)

$$(\tilde{\vec{\kappa}}_1 \cdot \vec{\partial}_{12})^{2m+1} (\tilde{\vec{\kappa}}_2 \cdot \vec{\partial}_{12})^{2n+1} \tilde{\eta}^{2(m+n)+1} = -\frac{[(2m+2n+1)!!]^2}{\tilde{\eta}} (\tilde{\vec{\kappa}}^2 - (\tilde{\vec{\kappa}} \cdot \hat{\eta})^2)^{m+n+1}, \qquad \text{(E1)}$$

and the identity Eq.(D5) in the form

$$\begin{aligned}
&\frac{\vec{\partial}_{12}\vec{\partial}_{12}}{\sqrt{m_1^2 + \tilde{\vec{\kappa}}^2}\sqrt{m_2^2 + \tilde{\vec{\kappa}}^2}} \Big(\frac{\tilde{\vec{\kappa}}}{\sqrt{m_1^2 + \tilde{\vec{\kappa}}^2}} \cdot \vec{\partial}_{12}\Big)^{2m+2} \Big(\frac{\tilde{\vec{\kappa}}}{\sqrt{m_2^2 + \tilde{\vec{\kappa}}^2}} \cdot \vec{\partial}_{12}\Big)^{2n} \frac{\tilde{\eta}^{2n+2m+3}}{(2m+2n+4)!} = \\
&= \frac{(2m+2n+3)!!(2m+2n+1)!!}{\tilde{\eta}(2m+2n+4)!(\sqrt{m_1^2 + \tilde{\vec{\kappa}}^2})^{2m+3}(\sqrt{m_2^2 + \tilde{\vec{\kappa}}^2})^{2n+1}} \Big[\mathbf{I}[\tilde{\vec{\kappa}}^2 - (\tilde{\vec{\kappa}} \cdot \hat{\eta})^2]^{m+n+1} + \\
&+ (2m+2n+1)[\tilde{\vec{\kappa}}\tilde{\vec{\kappa}} - (\tilde{\vec{\kappa}} \cdot \hat{\eta})(\tilde{\vec{\kappa}}\hat{\eta} + \hat{\eta}\tilde{\vec{\kappa}})](\tilde{\vec{\kappa}}^2 - (\tilde{\vec{\kappa}} \cdot \hat{\eta})^2)^{m+n} - \\
&- \hat{\eta}\hat{\eta}(\tilde{\vec{\kappa}}^2 - (\tilde{\vec{\kappa}} \cdot \hat{\eta})^2)^{m+n}(\tilde{\vec{\kappa}}^2 - (2m+2n+3)(\tilde{\vec{\kappa}} \cdot \hat{\eta})^2)\Big]. \qquad \text{(E2)}
\end{aligned}$$

the higher order Darwin plus spin part $V_{HDS} = U_{HDS} + U'_{HDS}$ becomes

$$\begin{aligned}
V_{HDS} &= -\sum_{i<j} \frac{Q_1 Q_2}{8\pi} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Big[\tilde{\vec{\mathbf{k}}}_1 \cdot \tilde{\vec{\mathbf{k}}}_2 \frac{[(2m+2n+1)!!]^2}{\tilde{\eta}(2n+2m+2)!} [\tilde{\vec{\kappa}}^2 - (\tilde{\vec{\kappa}} \cdot \hat{\eta})^2]^{n+m+1} \\
&\Big(\frac{1}{\sqrt{m_1^2 + \tilde{\vec{\kappa}}^2}}\Big)^{2m+1} \Big(\frac{1}{\sqrt{m_2^2 + \tilde{\vec{\kappa}}^2}}\Big)^{2n+1} \Big(\frac{1}{m_1^2 + \tilde{\vec{\kappa}}^2} + \frac{1}{m_2^2 + \tilde{\vec{\kappa}}^2}\Big) - \\
&- \frac{(2m+2n+3)!!(2m+2n+1)!!}{(2m+2n+4)!(\sqrt{m_1^2 + \tilde{\vec{\kappa}}^2})^{2m+1}(\sqrt{m_2^2 + \tilde{\vec{\kappa}}^2})^{2n+1}} \Big[\Big(\tilde{\vec{\mathbf{k}}}_1 \cdot \tilde{\vec{\mathbf{l}}}_2 [\tilde{\vec{\kappa}}^2 - (\tilde{\vec{\kappa}} \cdot \hat{\eta})^2]^{m+n+1} + \\
&+ (2m+2n+1)\Big(\tilde{\vec{\mathbf{k}}}_1 \cdot \tilde{\vec{\kappa}}\tilde{\vec{\mathbf{l}}}_2 \cdot \tilde{\vec{\kappa}} - \tilde{\vec{\mathbf{k}}}_1 \cdot (\tilde{\vec{\mathbf{l}}}_2 \cdot (\tilde{\vec{\kappa}}\hat{\eta} + \hat{\eta}\tilde{\vec{\kappa}}))(\tilde{\vec{\kappa}} \cdot \hat{\eta})\Big)(\tilde{\vec{\kappa}}^2 - (\tilde{\vec{\kappa}} \cdot \hat{\eta})^2)^{m+n} - \\
&- \tilde{\vec{\mathbf{k}}}_1 \cdot (\tilde{\vec{\mathbf{l}}}_2 \cdot \hat{\eta}\hat{\eta})(\tilde{\vec{\kappa}}^2 - (\tilde{\vec{\kappa}} \cdot \hat{\eta})^2)^{m+n}(\tilde{\vec{\kappa}}^2 - (2m+2n+3)(\tilde{\vec{\kappa}} \cdot \hat{\eta})^2)\Big) \\
&\Big(\frac{1}{m_1^2 + \tilde{\vec{\kappa}}^2} - \frac{1}{\sqrt{m_1^2 + \tilde{\vec{\kappa}}^2}\sqrt{m_2^2 + \tilde{\vec{\kappa}}^2}}\Big) + \\
&+ \Big(\tilde{\vec{\mathbf{k}}}_2 \cdot \tilde{\vec{\mathbf{l}}}_1 [\tilde{\vec{\kappa}}^2 - (\tilde{\vec{\kappa}} \cdot \hat{\eta})^2]^{m+n+1} + \\
&+ (2m+2n+1)\Big(\tilde{\vec{\mathbf{k}}}_2 \cdot \tilde{\vec{\kappa}}\tilde{\vec{\mathbf{l}}}_1 \cdot \tilde{\vec{\kappa}} - \tilde{\vec{\mathbf{k}}}_2 \cdot (\tilde{\vec{\mathbf{l}}}_1 \cdot (\tilde{\vec{\kappa}}\hat{\eta} + \hat{\eta}\tilde{\vec{\kappa}}))(\tilde{\vec{\kappa}} \cdot \hat{\eta})\Big)(\tilde{\vec{\kappa}}^2 - (\tilde{\vec{\kappa}} \cdot \hat{\eta})^2)^{m+n} - \\
&- \tilde{\vec{\mathbf{k}}}_2 (\tilde{\vec{\mathbf{l}}}_1 \cdot \hat{\eta}\hat{\eta})(\tilde{\vec{\kappa}}^2 - (\tilde{\vec{\kappa}} \cdot \hat{\eta})^2)^{m+n}(\tilde{\vec{\kappa}}^2 - (2m+2n+3)(\tilde{\vec{\kappa}} \cdot \hat{\eta})^2)\Big) \\
&\Big(\frac{1}{m_2^2 + \tilde{\vec{\kappa}}^2} - \frac{1}{\sqrt{m_1^2 + \tilde{\vec{\kappa}}^2}\sqrt{m_2^2 + \tilde{\vec{\kappa}}^2}}\Big) + \\
&+ \Big(\tilde{\vec{\mathbf{l}}}_2 \cdot \tilde{\vec{\mathbf{l}}}_1 [\tilde{\vec{\kappa}}^2 - (\tilde{\vec{\kappa}} \cdot \hat{\eta})^2]^{m+n+1} +
\end{aligned}$$

$$+ (2m+2n+1)\Big(\tilde{\vec{\mathbf{l}}}_2 \cdot \tilde{\vec{\kappa}}\tilde{\vec{\mathbf{l}}}_1 \cdot \tilde{\vec{\kappa}} - \tilde{\vec{\mathbf{l}}}_2 \cdot (\tilde{\vec{\mathbf{l}}}_1 \cdot (\tilde{\vec{\kappa}}\hat{\eta} + \hat{\eta}\tilde{\vec{\kappa}}))(\tilde{\vec{\kappa}} \cdot \hat{\eta})\Big)(\tilde{\vec{\kappa}}^2 - (\tilde{\vec{\kappa}} \cdot \hat{\eta})^2)^{m+n} -$$

$$- \tilde{\vec{\mathbf{l}}}_2 \cdot (\tilde{\vec{\mathbf{l}}}_1 \cdot \hat{\eta}\hat{\eta})(\tilde{\vec{\kappa}}^2 - (\tilde{\vec{\kappa}} \cdot \hat{\eta})^2)^{m+n}(\tilde{\vec{\kappa}}^2 - (2m+2n+3)(\tilde{\vec{\kappa}} \cdot \hat{\eta})^2)\Big)\Big(\frac{1}{\sqrt{m_1^2+\tilde{\vec{\kappa}}^2}\sqrt{m_2^2+\tilde{\vec{\kappa}}^2}}\Big)\Big]\frac{1}{\tilde{\eta}} +$$

$$+ \frac{[(2m+2n+1)!!]^2}{(2n+2m+2)!(\sqrt{m_1^2+\tilde{\vec{\kappa}}^2})^{2m+1}(\sqrt{m_2^2+\tilde{\vec{\kappa}}^2})^{2n+1}}$$

$$\Big(\tilde{\vec{\kappa}} \cdot \tilde{\vec{\mathbf{k}}}_2 \frac{(\tilde{\vec{\mathbf{k}}}_1 - \tilde{\vec{\mathbf{l}}}_1) \cdot \tilde{\vec{\kappa}} - (\tilde{\vec{\mathbf{k}}}_1 - \tilde{\vec{\mathbf{l}}}_1) \cdot \hat{\eta}\tilde{\vec{\kappa}} \cdot \hat{\eta}}{m_1^2+\tilde{\vec{\kappa}}^2} -$$

$$- \tilde{\vec{\kappa}} \cdot \tilde{\vec{\mathbf{k}}}_1 \frac{(\tilde{\vec{\mathbf{k}}}_2 - \tilde{\vec{\mathbf{l}}}_2) \cdot \tilde{\vec{\kappa}} - (\tilde{\vec{\mathbf{k}}}_2 - \tilde{\vec{\mathbf{l}}}_2) \cdot \hat{\eta}\tilde{\vec{\kappa}} \cdot \hat{\eta}}{m_2^2+\tilde{\vec{\kappa}}^2}\Big)\frac{(\tilde{\vec{\kappa}}^2 - (\tilde{\vec{\kappa}} \cdot \hat{\eta})^2)^{m+n}}{\tilde{\eta}} -$$

$$- \tilde{\vec{\mathbf{k}}}_1 \cdot \tilde{\vec{\mathbf{k}}}_2 \frac{[(2m+2n+3)!!]^2}{\tilde{\eta}(2n+2m+4)!}[\tilde{\vec{\kappa}}^2 - (\tilde{\vec{\kappa}} \cdot \hat{\eta})^2]^{n+m+2}\Big(\frac{1}{\sqrt{m_1^2+\tilde{\vec{\kappa}}^2}}\Big)^{2m+3}\Big(\frac{1}{\sqrt{m_2^2+\tilde{\vec{\kappa}}^2}}\Big)^{2n+3} +$$

$$+ \frac{(2m+2n+5)!!(2m+2n+3)!!}{(2m+2n+6)!(\sqrt{m_1^2+\tilde{\vec{\kappa}}^2})^{2m+3}(\sqrt{m_2^2+\tilde{\vec{\kappa}}^2})^{2n+3}}\Big(\tilde{\vec{\mathbf{k}}}_1 \cdot \tilde{\vec{\mathbf{k}}}_2[\tilde{\vec{\kappa}}^2 - (\tilde{\vec{\kappa}} \cdot \hat{\eta})^2]^{m+n+2} +$$

$$+ (2m+2n+3)\Big(\tilde{\vec{\mathbf{k}}}_1 \cdot \tilde{\vec{\kappa}}\tilde{\vec{\mathbf{k}}}_2 \cdot \tilde{\vec{\kappa}} - \tilde{\vec{\mathbf{k}}}_1\tilde{\vec{\mathbf{k}}}_2 \cdot \cdot (\tilde{\vec{\kappa}} \cdot \hat{\eta})(\tilde{\vec{\kappa}}\hat{\eta} + \hat{\eta}\tilde{\vec{\kappa}})\Big)(\tilde{\vec{\kappa}}^2 - (\tilde{\vec{\kappa}} \cdot \hat{\eta})^2)^{m+n+1} -$$

$$- \tilde{\vec{\mathbf{k}}}_1 \cdot (\tilde{\vec{\mathbf{k}}}_2 \cdot \hat{\eta}\hat{\eta})(\tilde{\vec{\kappa}}^2 - (\tilde{\vec{\kappa}} \cdot \hat{\eta})^2)^{m+n+1}(\tilde{\vec{\kappa}}^2 - (2m+2n+5)(\tilde{\vec{\kappa}} \cdot \hat{\eta})^2)\Big)\frac{1}{\tilde{\eta}}\Big]. \tag{E3}$$

We use

$$\frac{[(2m+2n+1)!!]^2}{(2n+2m+2)!} = \frac{(-)^{n+m}}{2(n+m+1)}\binom{-3/2}{n+m}, \tag{E4}$$

and we put $m+n=l$, so that $0 \leq m \leq l$ and $0 \leq l < \infty$. Then we perform the m sum using

$$\sum_{m=0}^{l}\left(\frac{x}{y}\right)^m = \frac{y^{l+1}-x^{l+1}}{y^l(y-x)}, \tag{E5}$$

and we obtain

$$V_{HDS} = -\sum_{i<j}\frac{Q_1Q_2}{8\pi}\sum_{l=0}^{\infty}\Big[[\tilde{\vec{\kappa}}^2 - (\tilde{\vec{\kappa}} \cdot \hat{\eta})^2]^l\Big(\frac{(\frac{1}{\sqrt{m_2^2+\tilde{\vec{\kappa}}^2}})^{2l+2} - (\frac{1}{\sqrt{m_1^2+\tilde{\vec{\kappa}}^2}})^{2l+2}}{m_1^2-m_2^2}\Big)\Big]$$

$$\Big(\tilde{\vec{\mathbf{k}}}_1 \cdot \tilde{\vec{\mathbf{k}}}_2 \frac{(-)^l}{2(l+1)}\binom{-3/2}{l}\frac{[\tilde{\vec{\kappa}}^2 - (\tilde{\vec{\kappa}} \cdot \hat{\eta})^2]}{\tilde{\eta}}\Big(\sqrt{\frac{m_2^2+\tilde{\vec{\kappa}}^2}{m_1^2+\tilde{\vec{\kappa}}^2}} + \sqrt{\frac{m_1^2+\tilde{\vec{\kappa}}^2}{m_2^2+\tilde{\vec{\kappa}}^2}}\Big) +$$

$$+\frac{(-)^l}{2(l+2)(2l+3)}\binom{-3/2}{l+1}\Big[\Big(\tilde{\vec{\mathbf{k}}}_1 \cdot \tilde{\vec{\mathbf{l}}}_2[\tilde{\vec{\kappa}}^2 - (\tilde{\vec{\kappa}} \cdot \hat{\eta})^2] +$$

$$+(2l+1)\Big(\tilde{\vec{\mathbf{k}}}_1 \cdot \tilde{\vec{\kappa}}\tilde{\vec{\mathbf{l}}}_2 \cdot \tilde{\vec{\kappa}} - \tilde{\vec{\mathbf{k}}}_1(\tilde{\vec{\mathbf{l}}}_2 \cdot (\tilde{\vec{\kappa}}\hat{\eta} + \hat{\eta}\tilde{\vec{\kappa}}))(\tilde{\vec{\kappa}} \cdot \hat{\eta})\Big) -$$

$$-\tilde{\vec{\mathbf{k}}}_1 \cdot (\tilde{\vec{\mathbf{l}}}_2 \cdot \hat{\eta}\hat{\eta})(\tilde{\vec{\kappa}}^2 - (2l+3)(\tilde{\vec{\kappa}} \cdot \hat{\eta})^2)\Big)\Big(\sqrt{\frac{m_2^2+\tilde{\vec{\kappa}}^2}{m_1^2+\tilde{\vec{\kappa}}^2}} - 1\Big) +$$

$$
\begin{aligned}
&+\Big(\tilde{\vec{\mathbf{k}}}_2\cdot\tilde{\vec{\mathbf{l}}}_1[\tilde{\vec{\kappa}}^2-(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2]+(2l+1)\Big(\tilde{\vec{\mathbf{k}}}_2\cdot\tilde{\vec{\kappa}}\tilde{\vec{\mathbf{l}}}_1\cdot\tilde{\vec{\kappa}}-\tilde{\vec{\mathbf{k}}}_2\tilde{\vec{\mathbf{l}}}_1\cdot\cdot(\tilde{\vec{\kappa}}\cdot\hat{\eta})(\tilde{\vec{\kappa}}\hat{\eta}+\hat{\eta}\tilde{\vec{\kappa}})\Big)-\\
&-\tilde{\vec{\mathbf{k}}}_2\cdot(\tilde{\vec{\mathbf{l}}}_1\cdot\hat{\eta}\hat{\eta})(\tilde{\vec{\kappa}}^2-(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2)^l(\tilde{\vec{\kappa}}^2-(2l+3)(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2)\Big)\Big(\sqrt{\frac{m_1^2+\tilde{\vec{\kappa}}^2}{m_2^2+\tilde{\vec{\kappa}}^2}}-1)+\\
&+\Big(\tilde{\vec{\mathbf{l}}}_2\cdot\tilde{\vec{\mathbf{l}}}_1[\tilde{\vec{\kappa}}^2-(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2]+(2l+1)\Big(\tilde{\vec{\mathbf{l}}}_2\cdot\tilde{\vec{\kappa}}\tilde{\vec{\mathbf{l}}}_1\cdot\tilde{\vec{\kappa}}-\tilde{\vec{\mathbf{l}}}_2\cdot(\tilde{\vec{\mathbf{l}}}_1\cdot(\tilde{\vec{\kappa}}\hat{\eta}+\hat{\eta}\tilde{\vec{\kappa}}))(\tilde{\vec{\kappa}}\cdot\hat{\eta})\Big)-\\
&-\tilde{\vec{\mathbf{l}}}_2\cdot(\tilde{\vec{\mathbf{l}}}_1\cdot\hat{\eta}\hat{\eta})(\tilde{\vec{\kappa}}^2-(2l+3)(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2)\Big)\Big]\Big)\frac{1}{\tilde{\eta}}+\\
&+\frac{(-)^l}{2(l+1)}\binom{-3/2}{l}\Big(\tilde{\vec{\kappa}}\cdot\tilde{\vec{\mathbf{k}}}_2\frac{(\tilde{\vec{\mathbf{k}}}_1-\tilde{\vec{\mathbf{l}}}_1)\cdot\tilde{\vec{\kappa}}-(\tilde{\vec{\mathbf{k}}}_1-\tilde{\vec{\mathbf{l}}}_1)\cdot\hat{\eta}\tilde{\vec{\kappa}}\cdot\hat{\eta}}{m_1^2+\tilde{\vec{\kappa}}^2}-\\
&-\tilde{\vec{\kappa}}\cdot\tilde{\vec{\mathbf{k}}}_1\frac{(\tilde{\vec{\mathbf{k}}}_2-\tilde{\vec{\mathbf{l}}}_2)\cdot\tilde{\vec{\kappa}}-(\tilde{\vec{\mathbf{k}}}_2-\tilde{\vec{\mathbf{l}}}_2)\cdot\hat{\eta}\tilde{\vec{\kappa}}\cdot\hat{\eta}}{m_2^2+\tilde{\vec{\kappa}}^2}\Big)\frac{\sqrt{m_1^2+\tilde{\vec{\kappa}}^2}\sqrt{m_2^2+\tilde{\vec{\kappa}}^2}}{\tilde{\eta}}+\\
&+\tilde{\vec{\mathbf{k}}}_1\cdot\tilde{\vec{\mathbf{k}}}_2\frac{(-)^l}{2(l+2)}\binom{-3/2}{l+1}\frac{[\tilde{\vec{\kappa}}^2-(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2]}{\tilde{\eta}\sqrt{m_1^2+\tilde{\vec{\kappa}}^2}\sqrt{m_2^2+\tilde{\vec{\kappa}}^2}}-\\
&-\frac{(-)^l}{2(l+3)(2l+5)}\binom{-3/2}{l+2}\Big(\tilde{\vec{\mathbf{k}}}_1\cdot\tilde{\vec{\mathbf{k}}}_2[\tilde{\vec{\kappa}}^2-(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2]^2+\\
&+(2l+3)\Big(\tilde{\vec{\mathbf{k}}}_1\cdot\tilde{\vec{\kappa}}\tilde{\vec{\mathbf{k}}}_2\cdot\tilde{\vec{\kappa}}-\tilde{\vec{\mathbf{k}}}_1\cdot(\tilde{\vec{\mathbf{k}}}_2\cdot(\tilde{\vec{\kappa}}\hat{\eta}+\hat{\eta}\tilde{\vec{\kappa}}))(\tilde{\vec{\kappa}}\cdot\hat{\eta})\Big)(\tilde{\vec{\kappa}}^2-(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2)-\\
&-\tilde{\vec{\mathbf{k}}}_1\cdot(\tilde{\vec{\mathbf{k}}}_2\cdot\hat{\eta}\hat{\eta})(\tilde{\vec{\kappa}}^2-(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2)(\tilde{\vec{\kappa}}^2-(2l+5)(\tilde{\vec{\kappa}}\cdot\hat{\eta})^2)\Big)\frac{1}{\sqrt{m_1^2+\tilde{\vec{\kappa}}^2}\sqrt{m_2^2+\tilde{\vec{\kappa}}^2}}\frac{1}{\tilde{\eta}}\Big]. \qquad \text{(E6)}
\end{aligned}
$$

We use

$$
\sum_{l=0}^{\infty}\frac{(-)^l}{2(l+1)}\binom{-3/2}{l}x^l=\frac{1}{x}((1-x)^{-1/2}-1)
$$

and similar identities applied in Appendix D and we obtain Eq.(8.3).

REFERENCES

[1] P.A.M. Dirac, "Lectures on Quantum Mechanics", Belfer Graduate School of Science, Monographs Series (Yeshiva University, New York, 1964).

[2] J.L.Anderson and P.G.Bergmann, Phys.Rev. **83**, 1018 (1951). P.G.Bergmann and J.Goldberg, Phys. Rev. **98**, 531 (1955).

[3] For a recent text book discussion of Hamiltonian constraints see S. Weinberg,"The Theory of Fields" Volume 1, (Sec 7.6) (Cambridge Univ. Press, Cambridge, 1995)

[4] L.Lusanna, Int.J.Mod.Phys. **A12**, 645 (1997).

[5] H.Crater and L.Lusanna, "The Rest-Frame Darwin Potential from the Lienard-Wiechert Solution in the Radiation Gauge", to appear in Annals of Physics (HEP-TH/0001046).

[6] L.Lusanna, "Towards a Unified Description of the Four Interactions in Terms of Dirac-Bergmann Observables", invited contribution to the book "Quantum Field Theory, A 20th Century Profile"of the Indian National Science Academy for the International Mathematics Year 2000 AD, ed. A.N.Mitra, foreword F.J.Dyson (Hindustan Book Agency, 2000)(HEP-TH/9907081). "Tetrad Gravity and Dirac's Observables", talk given at the Conf. "Constraint Dynamics and Quantum Gravity 99", Villasimius 1999, ed. V.DeAlfaro et al. Nucl.Phys. (Proc. Suppl.) **B88**, 301 (2000) (GR-QC/9912091)."The Rest-Frame Instant Form of Dynamics and Dirac's Observables", talk given at the Int.Workshop"Physical Variables in Gauge Theories", Dubna 1999 (HEP-TH/9912203).

[7] D.Alba and L.Lusanna, Int.J.Mod.Phys. **A13**, 3275 (1998) (HEP-TH/9705156).

[8] A.Schild, Phys.Rev. **131**, 2762 (1963). A.Schild and J.A.Schlosser,J.Math.Phys. **9**, 913 (1968). R.A.Rudd and R.N.Hill, J.Math.Phys. **11**, 2704 (1970). C.M.Andersen and H.C.von Baeyer, Ann.Phys.(N.Y.) **60**, 67 (1970) and Phys.Rev. **D5**, 802 (1972). P.Cordero and G.C.Ghirardi, J.Math.Phys. **14**, 815 (1973). A.Schild, Ann.Phys. (N.Y.) **93**, 88 (1975) and **99**, 434 (1976). P.Stephas, Am.J.Phys. **46**, 360 (1978). P.Palou, J.Stela and L.Mas, J.Math.Phys. **26**, 2192 (1985).

[9] F.Bigazzi and L.Lusanna, Int.J.Mod.Phys. **A14**, 1429 (1999) (HEP-TH/9807052).

[10] L.Lusanna and M.Materassi, Int.J.Mod.Phys. **A15**, 2821 (2000) (HEP-TH/9904202).

[11] A.Barducci, R.Casalbuoni and L.Lusanna, Nuovo Cimento **A35**, 377 (1976); Lett.Nuovo Cim. **19**, 581 (1977).

[12] B.Thaller, "The Dirac Equation" (Springer, Berlin, 1992).

[13] A. Barducci, R. Casalbuoni and L. Lusanna, Phys. Lett. **B64**,319 (1976)

[14] H.A.Bethe and E.E.Salpeter, "Quantum Mechanics of One and Two Electron Atoms" (Academic Press, New York, 1957) p 193.

[15] T.P.Das, "Relativistic Quantum Mechanics of Electrons" (Harper and Row, New York, 1973).

[16] G.Breit, Phys. Rev. **34**, 553 (1929)

[17] P. Van Alstine and H.W. Crater, Phys. Rev D15 **33** 1037, (1986)

[18] M.A.Stroscio, Phys.Rept. **22**, 215 (1975).

[19] A.Barducci, R.Casalbuoni and L.Lusanna, Nucl.Phys. **B124**, 93 (1977).

[20] D.Alba and L.Lusanna, Int.J.Mod.Phys. **A13**, 2791 (1998).

[21] D.Alba, L.Lusanna and M.Pauri, "Centers of Mass and Rotational Kinematics for the Relativistic N-Body Problem in the Rest-Frame Instant Form", Firenze Univ. preprint 2001 (HEP-TH/0102087).

[22] L.Lusanna, Int.J.Mod.Phys. **A10**, 3531, 3675 (1995).
[23] C.Lämmerzahl, J.Math.Phys. **34**, 3918 (1993).
[24] J.M.Pons and L.Shepley, Class.Quantum Grav. **12**, 1771 (1995)(GR-QC/9508052); J.M.Pons, D.C.Salisbury and L.C.Shepley, Phys. Rev. **D55**, 658 (1997)(GR-QC/9612037).
[25] A.Barducci, R.Casalbuoni and L.Lusanna, Lett.Nuovo Cimento **19**, 581 (1977).
[26] A.Papapetrou, Proc.Roy.Soc.London **A209**, 248 (1951).
[27] A.Barducci and L.Lusanna, Nuovo Cimento **47B**, 54 (1978); **53B**, 59 (1979).
[28] A.Barducci, R.Casalbuoni and L.Lusanna, Nuovo Cimento **54A**, 340 (1979).
[29] V.Barmann, L.Michel and V.L.Telegdi, Phys.Rev.Letters **2**, 435 (1959), Nucl.Phys. **B124**, 93 (1977).
[30] A.Wintner, "The Analytical Foundations of Celestial Mechanics" (Princeton Univ.Press, Princeton, 1947). W.M.Smart,"Celestial Mechanics" (Longmans, London, 1953).
[31] A.N.Gordeyev, Theor.Math.Phys. **36**, 593 (1978).
[32] Ch.G.van Weert, J.Phys. **A9**, 477 (1976).
[33] W.Drechsler and A.Rosenblum, Phys.Lett. **B106**, 81 (1981). W.Drechsler, P.Havas and A.Rosenblum, Phys.Rev. **D29**, 658, 668 (1984).
[34] K.Kuchar, J.Math.Phys. **17**, 777, 792 and 801 (1976).
[35] P.A.M.Dirac, *Rev.Mod.Phys.* **21** (1949) 39
[36] Ph.Droz Vincent, Lett.Nuovo Cimento **1**, 839 (1969); Phys.Scr. **2**, 129 (1970); Ann.Inst.H.Poincaré **27**, 407 (1977) and **32A**, 377 (1980); Phys.Rev. **D19**, 702 (1979).
[37] I.T.Todorov,
JINR Report No.E2-10175, Dubna, 1976 (unpublished); Ann.Inst.H.Poincaré **28**, 207 (1978); "Constrainy Hamiltonian Approach to Relativistic Point Particle Dynamics I", SISSA preprint, Trieste 1980; "Constraint Hamiltonian Mechanics of Directly Interacting Relativistic Particles" in "Relativistic Action at a Distance: Classical and Quantum Aspects", ed.J.Llosa, Lecture Notes Phys. 162 (Springer, Berlin, 1982).
[38] P. Van Alstine, "Relativistic Two Body Problem from Singular Actions", Ph.D. Dissertation Yale University, 1976; M.Kalb and P. Van Alstine, Yale Reports Nos. C00-3075-146, 1976 and C00-3075-156,1976.
[39] G.Longhi and L.Lusanna, Phys.Rev. **D34**, 3707 (1986).
[40] H.W.Crater and P.Van Alstine, J.Math.Phys. **23**, 1997 (1982) ; Ann.Phys.(N.Y.) **148**, 57 (1983); Phys.Rev. **D30**, 2585 (1984), **D33**, 1037 (1086), **D36**, 3007 (1987) and **D46**, 766 (1992). H.W.Crater and D.Yang, J.Math.Phys. **32**, 2374 (1991).
[41] H.Sazdjian, Phys.Rev.Lett. **156B**, 381 (1985); Phys.Rev. **D33**, 3401 (1986); J.Math.Phys. **28**, 2618 (1987); Ann.Phys.(N.Y.) **191**, 52 (1989);"Connection of Constraint Dynamics with the Bethe-Salpeter Equation", talk at the Int.Symposium "Extended Objects and Bound States", Karuizawa 1992, eds. O.Hara, S.Ishida and S.Naka (World Scientific, Singapore, 1992).
[42] H.W.Crater and P.Van Alstine, Phys.Rev.Lett. **53**, 1577 (1984); J.Math.Phys. **31**, 1998 (1990); Phys.Rev. **D30**, 2585 (1984); **D34**, 1932 (1986); **D36**, 3007 (1987); **D37**, 1982 (1988). H.W.Crater, R.L.Becker, C.Y.Wong and P.Van Alstine, Phys.Rev **D46** , 5117 (1992). H.W.Crater, C.W.Wong and C.Y.Wong, Found. Phys. **24**, 297 (1994). H.Crater and P.Van Alstine, in "Constraint's Theory and Relativistic Dynamics", Firenze 1986, eds. G.Longhi and L.Lusanna (World Scientific, Singapore, 1987) and in "Constraint

Theory and Quantization Methods", Montepulciano 1993, eds. F.Colomo, L.Lusanna and G.Marmo (World Scientific, Singapore, 1994).

[43] H.Sazdjian, Phys.Rev. **D33**, 3425 and 3435 (1986); J.Math.Phys. **29**, 1620 (1988). M.Bawin, J.Cugnon and H.Sazdjian, Int.J.Mod.Phys. **A9**, 5711 (1994); **A11**,5303 (1996). J.Mourad and H.Sazdjian, J.Math.Phys. **35**, 6379 (1994); J.Phys. **G21**, 267 (1995). H.Jallouli and H.Sazdjian, Phys.Lett. **B366**, 409 (1996).

[44] F.Rohrlich and L.P.Horwitz, Phys.Rev. **D24**, 1528 (1981). H.Sazdjian, Nucl.Phys. **B161**, 469 (1979).

[45] H.W.Crater and D.Yang, J.Math.Phys. **32**, 2374 (1991)

[46] X. Jaen, J. Llosa, and A. Molina, Phys. Rev. **D34**, 2302 (1986)

[47] R. A. Moore and T. C. Scott, Phys. Rev. **A46,** 3637 (1992)

[48] V.N Golubenkov and A. Ia. Smorodinski, Sov. Phys. JETP **31**, 330 (1956)

[49] I.Herbst, Commun.Math.Phys. **53**, 285 (1977); **55**, 316 (1997). B. and L. Durand, Phys.Rev. **D28**, 396 (1983); erratum Phys.Rev. **D50**, 6642 (1994). J.J.Basdevant and S.Boukraa, Z.Phys. **C28**, 413 (1985).A.Martin and S.M.Roy, Phys.Lett. **B233**, 407 (1989).A.LeYaouanc, L.Oliver and J.C.Raynal, Ann.Phys.(N.Y.) **239**, 243 (1995).
W.Lucha and F.F.Schöberl, Phys.Rev. **D50**, 5443 (1994); "Spinless Salpeter Equation: Analytic Results", talk at the XI Int.Conf.: Problems of Quantum Field Theory, Dubna 1998 (HEP-PH/9807342); Int.J.Mod.Phys. **A14**, 2309 (1999).